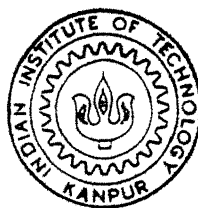


# ANALYSIS AND DETERMINATION OF THE SOURCE TERM IN COUPL REACTION—DIFFUSION AND HEAT CONDUCTION PROBLEMS

by  
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DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY KANPUR  
MAY, 1994

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ANALYSIS AND DETERMINATION OF THE SOURCE TERM IN COUPLED  
REACTION-DIFFUSION AND HEAT CONDUCTION PROBLEMS

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DOCTOR OF PHILOSOPHY

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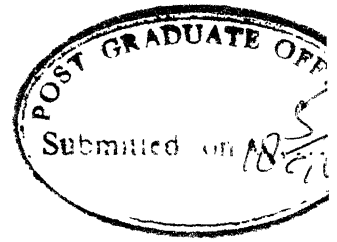
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CERTIFICATE

Certified that the work presented in this thesis entitled "ANALYSIS AND DETERMINATION OF THE SOURCE TERM IN COUPLED REACTION-DIFFUSION AND HEAT CONDUCTION PROBLEMS" by ARATI NANDA has been carried out under my supervision and has not been submitted elsewhere for a degree.

(Professor P. C. Das)  
Department of Mathematics  
Indian Institute of Technology, Kanpur.

May, 1994.



Dedicated to

MANTU

&

MY PARENTS



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May, 1994.

Arati Nanda



## ABSTRACT

In recent years due to requirements of modern technology and advances in computing, there is a considerable interest in a wide variety of inverse problems. These are basically IBVP incomplete in the normal sense and ill-posed from that point of view. These problems are solved with the help of additional observed data.

The thesis addresses itself to two such problems. It consists of five chapters. In the first chapter we give a brief introduction to literature on the subject. The basic content of the thesis consists of two blocks of two chapters each. In chapters two and three the inverse problem called the source problem for coupled Reaction-Diffusion system is considered, whereas in the second block i. e., chapters four and five, space variable-dependent-source-problem for the Heat Conduction problem is analyzed.

In chapter two, the coupled Reaction-Diffusion system in a semi-infinite spatial domain is studied. The basic results consist an iterative algorithm for construction of the reaction function and showing the existence of such a function in Hölder function space. Towards that end the problem is reformulated as a fixed point problem of a certain operator. Using the celebrated SCHAUDER's fixed point theorem, the existence of the original problem is established.

In chapter three, under additional hypotheses, we have established the uniqueness of the above solution by the method of center contraction mapping property of the operator and there by simultaneously showing the convergence of the iterative scheme.

The second block of results consist of determination of a source function depending on the spatial variable in addition to the unknown solution variable in a bounded spatial domain.

In chapter four we have analyzed the existence of the source function and in chapter five the uniqueness and the convergence property of the iteration scheme are established.



## SYNOPSIS

# ANALYSIS AND DETERMINATION OF THE SOURCE TERM IN COUPLED REACTION-DIFFUSION AND HEAT CONDUCTION PROBLEMS

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KANPUR-208016, MAY, 1994

In recent years due to requirements of modern technology and advances in computing, there is a considerable interest in a wide variety of inverse problems. Most of these arise in situations where certain basic structure of physical models are assumed and observed data are required to be used to concretize the models. It is usually done either by determining unknown parameters involved in the model or determining unknown dependence, expressed as functions of certain physical entities such as temperature, density or the rate of reaction in a chemical process. The source of inverse problems could be as diverse as medicine, geophysics or space technology.

There are many types of inverse problems involving PDE. These are basically IBVP with incomplete information such as "lack of information about certain parameters" or "the absence of knowledge about certain input functions". Determination of source term or the input function, is called the source problem. While solving inverse problems one faces the lack of information which has to be compensated by additional supply of observed or observable data. As one would expect, these problems are generally ill-posed and difficult from the view of analysis, because of generally nonlinear nature of the problems.

Several people have worked on source problems. Indeed, assuming the monotonicity of certain class of overposed data, DuChateau and Rundell [1] have determined the unknown reaction term in a  $n$ -dimensional single R-D equation. An analogous result in  $R^1$  using a different approach was obtained by Muzylev in [2]. Although in [1] and [2] the uniqueness of the solution of the above source problems have been shown in the case when the forcing function is assumed to be analytic, the question of the existence was not answered. However, Pilant and Rundell in [3] have shown the existence and uniqueness to the above problem for a certain class of functions in a small time interval. In [4] Pilant and Rundell and in [5] Anger have studied the same problem by reformulating it as a fixed point problem. The present work possibly the first work for a coupled system of two equations dealing with reaction diffusion system derives results analogous to those by Pilant and Rundell obtained for the one dimensional heat conduction equation. Later on, source problem for heat conduction equation, with the source function depending on space variable is considered, where the more complex case of bounded spatial region is the underlying domain.

The thesis consists of five chapters. In the first chapter we give a brief introduction to literature on the subject along with an outline of the contents of different chapters. The basic content of the thesis consists of two blocks of two chapters each. In chapters two and three the source problem for coupled Reaction-Diffusion problem is considered, whereas in the second block i. e., chapters four and five, space variable-dependent-source-problem for the Heat Conduction problem is analyzed.

Now we summarize the basic results contained in the first block i. e., chapters II and III.

It is well known that Reaction-Diffusion equations arise in most chemical processes including the problems of flame propagation. One of the

major problems is to assess the nature of the reaction. For example, problems of one space dimension, the equations generally take the form

$$u_t - u_{xx} = f(u)g(v) \quad ; \quad v_t - v_{xx} = -f(u)g(v),$$

where  $f$  and  $g$  are to be determined. From physical considerations one of functions " $g$ " is known analytically. Therefore, the source problem consists the determination of  $f$  from experimental data. Quite often, in practice the structure of  $f$  is assumed involving certain parameters. Then using some ad hoc procedures like assignment of values to these parameters and comparing with experimental data, one is led to a set of parametric values for which the experimental results are very close. However, in the present work we have given an analytical and constructive way to determine such an  $f$  without assuming explicit structure of  $f$  in terms of some parameters.

In chapter two, we have considered the following mathematical model in semi-infinite domain  $(0, \infty)$  and  $t > 0$ .

$$\begin{aligned} (A) \quad & \begin{aligned} u_t - u_{xx} &= f(u)g(v) & ; & & v_t - v_{xx} &= -f(u)g(v) \\ u(x, 0) &= u_0(x) & ; & & v(x, 0) &= v_0(x) \\ u_x(0, t) &= g_1(t) & ; & & v_x(0, t) &= g_2(t) \end{aligned} \end{aligned}$$

with superposed known data ;  $u(0, t) = \theta(t)$ .

Then our quest is to determine  $f$  assuming  $g$  to be a known function of  $v$  such that the overposed condition is satisfied. The basic results of this chapter are as follows :

#### BASIC RESULTS :

- (1) Presenting an iterative algorithm for construction of  $f$ .
- (2) Showing the existence of such an " $f$ " in a certain function class.

The basic algorithm consists of a Picard type function evaluation scheme. If  $f^{(n)}$  is a given function then determine  $u^{(n)}$ ,  $v^{(n)}$  from the direct problem (A) which will not in general satisfy the overposed Dirichlet data at the boundary. However, at the boundary  $x = 0$ , we need to satisfy the following relation

$$\theta' - u_{xx} = f(\theta) g(v),$$

which is not valid for arbitrary choice of  $f$ . Thus the following updating technique is used i.e., knowing  $f^{(n)}$ ,  $u^{(n)}$ ,  $v^{(n)}$ , one obtains a new function  $f^{(n+1)}$  satisfying

$$\theta' - u_{xx}^{(n)} = f^{(n+1)}(\theta) g(v^{(n)}).$$

This inspires the introduction of the following operator

$$T_{\theta}[f](t) = \frac{\theta'(t) - u_{xx}(0,t;f,g)}{g(v(0,t;f,g))}.$$

Then the iteration scheme is given by

$$f^{(n+1)}(\theta(t)) = T_{\theta}[f^{(n)}](t).$$

It is shown that the existence of the solution of the inverse problem is equivalent to the existence of a fixed point of  $T_{\theta}$  in  $C^{\alpha}$ ,  $0 < \alpha < 1$ . The self mapping property of  $T_{\theta}$  is established in a bounded ball in  $C^{\alpha}$  and using the compact imbedding in the scales of  $C^{\alpha}$  the existence is established.

Since  $T_{\theta}$  involves  $u_{xx}$ , it is difficult to obtain different properties of the operator, therefore, it is expressed in an integral form with respect to  $u$  which measures the nonlinearity of the R. H. S. of (A). By using the Green's function, the solution  $u$  is expressed in terms of integral expression which is much easier to handle. The properties of Green's function are used to give the required estimates in  $C^{\alpha}$  for  $0 < \alpha < 1$ . The actual bounds are calculated explicitly and using the property of the identity map to be a

compact imbedding from  $C^\alpha \rightarrow C^\beta$  for  $\beta < \alpha$ , the existence of a fixed point of the operator is shown.

In chapter three, we have studied the uniqueness of the above solution and have also shown the convergence of the iteration map. Extra smoothness property on both  $f$  and  $g$  ensures certain estimates which lead to the required conclusions. The whole process involves a long list of estimates.

The second block of results consist of determination of a source function depending on the spatial variable in addition to the unknown solution variable. The domain considered here is a bounded domain. Pilant and Rundell have considered the heat equation of the following type in a semi-infinite region.

$$u_t - u_{xx} = F_1(u),$$

where the source term is a function of  $u$  only. Clearly, more general form of  $F_1$  involving spatial variable is of the type  $F_1(u, x)$ . In the thesis we have considered  $F_1(u, x)$  where variation of  $F_1$  with respect to  $x$  is allowed up to linear terms in " $x$ ". Then the equation can be written as

$$u_t - u_{xx} = F_1(u) + x F_2(u),$$

which can be rewritten as

$$u_t - u_{xx} = (1 - x) f_1(u) + x f_2(u) + \gamma(x, t)$$

in a domain  $(0, 1) \times \mathbb{R}^+$ . Besides, we assume the following Initial Boundary conditions.

$$u(0, t) = u(x)$$

$$u_x(0, t) = g_1(t) \quad , \quad u_x(1, t) = g_2(t).$$

Since there are two unknown functions to be determined, we require two overposed Dirichlet data which we give at the two boundary points i. e.,

$$u(0, t) = \theta(t) \quad \text{and} \quad u(1, t) = q(t).$$

In chapter four we have analyzed the existence of the solution  $(u, f_1, f_2)$  and in chapter five the uniqueness and the convergence property of the iteration scheme. Here the Green's function is an infinite series and the underlying function space is  $C^\alpha$  with  $0 < \alpha < 1/2$ .

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## CHAPTER I

### INTRODUCTION

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#### 1.1 MOTIVATION :

The modern inverse problems will date back to N. H. ABEL, whose pioneering work in 1823 led to an integral equation of the first kind (KELLER 1976 [76], ANGER, 1990 [7]). His original problem was to determine the shape of a hill from the time a particle takes to return when it slides up on a frictionless hill with certain initial energy. Questions regarding uniqueness of the solution to this problem were answered by KELLER (1976) [76] and ANGER (1987) [7]. In course of time more several inverse problems were posed in the connection as well as in connection with investigation of the interior of the obstacle. Some of them turn out to be very important from the point of view of modern technology. A large number of inverse problems are motivated by the desire to know the past states of physical system, or determination of unknown parameters involved in a physical model or to determine certain parameters in order to influence a system to behave in a desired manner in future.

#### 1.2 INVERSE PROBLEMS :

Physical problems in different fields of Science, Technology and Medicine often give rise to mathematical models of inverse problems. Several practically important models involve differential equations with insufficient

informations. These deal with determination of unknown coefficients, boundary or initial conditions or source terms from observed or experimental overdetermined data. The Cauchy problems for a parabolic one in backward time direction and Cauchy problems for hyperbolic equation with data on a time like manifold (JOHN, 1955 [73]) are some classical examples.

### 1.3 ASSOCIATED DIFFICULTIES :

The first question that arises in the inverse problem is the physical wholesomeness of the proposed model. If it is not so, then it is one of the many finite models that may satisfy the observed data. The reasons for this can be traced to (1) inherent nonuniqueness; for example potential fields are inherently nonunique (2) uncertainty in data; caused by experimental error and insufficient model specification or the mixture of these two. For example the information provided by the finite number of sensors inside a heat conducting body are at discrete locations and measured at discrete time interval results in incomplete data of spatial temperature variation.

Another difficulty associated with the heat conduction problems in particular is, the temperature response of an internal point is quite different from that of a point at the surface. These are some of the demanding aspects encountered in inverse problems.

### 1.4 SCOPE OF INVERSE PROBLEMS :

It is easily observed how inverse problems arise in the following areas of scientific activity.

#### (1) EARTH SCIENCE :

The principal task of geophysics is to study the internal structure of the earth based on surface or sub surface observations. The shortage of the information on gravitational field inside the earth results in uncertainty and

nonuniqueness of solutions. But the first uniqueness results (for special mass distribution) were obtained by G. HERGLOTZ (1914) and P. S. NOVIKOV (1938). The spherical symmetrical model of the earth played a major role in knowledge of the internal structure such as crust, coating and nucleus. The coupling of different fields in geophysics i.e., coupling of gravitational and seismic fields (A. S. ALEKSEEV, B. A. BUBNOV (1981,1984)) as well as coupling of gravitational and magnetic fields (D. ZIDAROV (1968-1984), M. S. ZHDANOV (1981, 1988)) are of particular interest. In addition, satellites introduced the possibility of measuring various kinds of radiation. Associated with gravitational and magnetic surveying is one more inverse problem dealing with the theory of potential. It is used in geophysics for locating the deposits of mineral resources via gravity anomalies. The first theorem of uniqueness for an inverse problem of the Newton's potential was formulated by P. S. NOVIKOV in 1938.

## (2) SCATTERING THEORY :

The inverse problem of scattering theory originates in the theoretical physics. The problem here is of determination of the parameters of sources, diffusor or propagation media by means of the radiation received by a detector (ANGER, 1990 [7]). The first uniqueness result in inverse scattering theory related to Sturm-Liouville operator was formulated and investigated by V. A. AMBARTSUMIAN in 1929. Since 1964, various forms of inverse problems have been studied by G. BERG, N. LEVINSON, V. A. MARCHENKO, I. M. GELFAND and B. M. LEVITAN etc.. FADDEYEV (1959) [45] determined a medium for which an electromagnetic wave is reflected knowing the reflection coefficient. Similar questions are also asked in quantum mechanics. The paper translated by SECKLER (FADDEYEV, 1959) [45] gives a nice review on the inverse scattering theory. The early book of AGRANOVICH and MARCHENKO (1963) [1] discusses the variants

of inverse problem arising in connection with the quantum theory of scattering which is apparently the most interesting from the stand point of application.

### 3) MEDICINE : (COMPUTER-AIDED TOMOGRAPHY)

Computer-aided tomography in medicine consists of the reconstruction of a function from their projection along lines (NATTERER, 1986) [95]. "Tomography" is derived from the greek word meaning slice. In the early 1970s CT was introduced in diagnostic radiology. Complete bibliography with excellent survey on the applications of CT is cited in (DEANS, 1983) [37].

### 1.5 ILL-POSEDNESS :

Mathematical physics frequently give rise to ABEL's integral equation which is of first kind. The small structural difference between the first and second kind of Fredholm integral equations changes the whole theory drastically. The Integral equations of second kind are wellposed whereas the equations of first kind are inherently unstable. The smoother the kernel, the greater the risk to handle such equations.

In the beginning of the twentieth century J. HADAMARD (1923) defined "well-posedness" of a problem in a classical sense. A problem is called wellposed if the operator equation corresponding to it has a unique continuously dependent solution. If there is any deficiency as regards any of the above three conditions, the problem is called illposed or incorrectly posed in the sense of HADAMARD.

A closer view of HADAMARD's definition ensures that the wellposedness of the operator equation is intimately connected not only with the operator but also with the spaces and the topology they carry i. e., it is the property of the triplet, space, operator and topology.

For the problems which are not correct in the classical sense TIKHONOV suggested a new notion of correctness (TIKHONOV, 1963) [118]. His

regularization method is based on the celebrated Riemann-Lebesgue lemma. The role of the regularizing parameter is proposed to damp out the oscillations of the solution.

Almost all works on Inverse Problems discuss about the illposed nature of the inverse problems. ROMANOV (1987) [106] and DONTCHEV (1993) [39] compare Tikhonov Regularization with the original classical notion of HADAMARD wellposedness. They give a nice distinction between the two. Further they illustrate that by modifying the definition of stability and assuming a priori the existence of the solution, one can overcome the rigidity of the classical wellposedness.

The mathematical tool developed by TIKHONOV's conventional regularization has by now a wide variety of applications. DEMRI (1992) [38] discusses different aspects of this method which are proved to be applicable for geophysical problems. In another classification the regularization procedures have been classified into three categories. (1) Tikhonov regularization (2) Constrained least squares regularization and (3) Wiener Filter Regularization. From computational point of view regularization methods are grouped together as (1) Sequential regularization method (BECK 1985) [18] (2) Trial function regularization method (3) Zeroth order regularization method (4) Generalized Sequential function specification regularization method. For more information see DEMRI (1992) [38] and the literature cited there.

## 1.6 METHODS :

Every inverse problem requires a particular approach since no single approach is suitable for all types of problems. The methods are generally designed to solve a particular category of problem, each category being defined by the relationship between the perturbation of the model and its effects on the observations. In several engineering contexts, it is sometimes

necessary to estimate the surface temperature in a body from a measured history at a fixed location inside the body. The corresponding inverse thermal problem may be called internal in LAVRENT'EV's terminology (KLIVANOV, 1985) [80].

Various solution methods have been applied to the inverse problems including (1) Integral equation solution (2) Series solution (3) Transform solutions and (4) Function minimization technique.

One of the earliest papers on the IHCP (Inverse Heat Conduction Problem) was published by STOLZ in 1960 [113], where he calculated the heat transfer rates during quenching of simple finite shapes. He obtained a linear solution by numerical inversion of the integral solution of the direct problem. His result was found to be unstable for small time steps.

FICHERA (1961) [46] succeeded in extending the single layer theory to Dirichlet problems for higher order strongly elliptic differential equations in two independent variables. The corresponding integral equations derived from the boundary conditions have logarithmic Kernel, rather than Cauchy kernels which were used in the classical approach. This replacement has the advantage of allowing simpler numerical computations and obtaining optimal rate of convergence (HSIAO-WENDLAND, 1977) [66].

BURGGRAFF (1964) [21] devised a series solution to linear inverse problem which is exact only for continuous output data. An important drawback of this form is that they contain derivatives of arbitrary orders with respect to time of the experimentally determined temperature and its gradient at points inside the body. This immediately limits the applicability of series constructed in this way because of the unreliability of the values of the derivatives. In TSIREL'MAN, 1985) [121] the author has demonstrated the construction of an analytical solution using Laplace transform which is free from this difficulty and correct within the class of analytical functions. IMBER (1974) [67] has

developed a transform solution for two dimensional bodies of arbitrary shape. Details can be found in (WOO, 1981) [127].

BECK (1968) [16] utilized a least squares technique to generate solution for a much smaller time step using the integral approach similar to STOLZ. (ARLEDGE et al. 1977) [8] the authors used an integral solution approach which is valid for constant thermal properties.

The finite difference and finite element methods have been the predominant numerical techniques for solution of the direct problem of heat conduction and they have been applied to non-linear inverse formulation by O (1975) [96], BECK (1970) [17], BASS (1980) [15] and YOSHIMURA (1985) [128].

In (TALENTI-VESSELLA, 1982) [117] the authors have presented the stability estimates of the heat equation. The problem satisfying some growth condition restores stability of the solutions. MIKHAILOV (1983, 1985) [89, 90] in an external formulation investigates the convergence of iterative methods. These methods are based on the search for boundary functions starting from the requirement of minimization of a certain functional. Iteration approximations to the desired function are constructed by the conjugate gradient method.

KEROV (1983) [77] solved the heat conduction problem in a cylindrical coordinate system where as in 1985, ALNAJEM and OZISIK [5] solved it for three dimensional problem using the least square technique to compute the unknown parameters associated with solution.

BEN-HAIM and ELIAS (1987) [20] measured the surface temperature and heat flux by optimal design using convexity analysis which is developed by BEN-HAIM in 1985 [19].

ALIFANOV solved the IHCP by iterative methods in 1974 [2]. But he found that there is no uniform convergence in the calculation of the gradient and the accuracy of the solution of IHCP depends to a considerable extent on the

selection of the initial approximation. But MIKHAILOV (1983) [89] modified the procedure slightly and got a better result. The iteration scheme based on searching for the surface temperature is not only uniformly convergent but also assures the greatest accuracy in restoring the heat flux density. A number of methods for solving inverse heat conduction problems are analyzed in (ALIFANOV, 1983) [3] from the point of view of their practical use.

Based on the use of gradient of the error functional and the functional deficiency gradient for the iterative solution of inverse parabolic problem various formulations of nonlinear inverse problems of generalized heat conduction were discussed in (ALIFANOV, 1987) [4].

Parametric optimization methods are analyzed in (VIGAK, 1986) [123], (MATSEVITY, 1988) [88]. In (VIGAK, 1986) [123] the authors have elaborated the result of VIGAK (1983) [122] and proposed a method to solve a control problem for a one-dimensional temperature system.

MURIO (1988) [93], introduces a new automatic algorithm to uniquely determine the radius of mollification as a function of the amount of noise in the data. This parameter selection criterion is a very important practical detail when attempting to solve a real problem and leads naturally to a simple and powerful computational technique.

BANKS AND WADE (1989) [14], [124] have introduced weak Tau method for parameter determination problems.

INGHAM (1992) [68] used BEM to solve the inverse nonlinear problem.

For internal measurement please refer to (CARASSO, 1982) [33], (KLIBANOV, 1985) [80], (MUZYLEV, 1985) [94], (WEBER, 1981) [125], (IMBER, 1974) [67], (HORE, 1977) [63], (KEROV, 1983) [77].

## 1.7 COEFFICIENT PROBLEMS :

### (1) DISTRIBUTED PARAMETER SYSTEM :

The measurement problem in distributed system is posed as observability question. Sensor location and the information content of resulting signals relative to a partial differential equation model are primary questions. New definition of observability has been introduced for purpose by the authors GOODSON and KLEIN (1970) [54]. Indeed the above definition states observability as a uniqueness question and does not consider the subsequent estimation problem. However, the definition is independent of particular analytical solution form. Therefore, the full range of classic and modern mathematical techniques may be employed to answer observability questions.

### (2) PARABOLIC PROBLEMS :

The coefficient problem for heat operator was formulated by JONES JR. in 1962 [74]. He determined the conductivity of a medium if it was known a priori to be a function of time only. It was shown that the problem is equivalent to a certain nonlinear integral equation for the coefficient and both possess unique solution simultaneously. The constructive existence proof suggested by KELLER is utilized to find a numerical approximation to the solution (DOUGLAS 1962) [40].

In (JONES JR., 1963) [75] the author solves five problems using the integral method equation developed in 1962. The unknown coefficient is assumed to be positive and continuous to establish the results. For special cases explicit formula has given.

CANNON (1963) [22] considered linear transformation with respect to data and the integral transformation for the coefficient to yield a simple analysis of the existence and uniqueness of JONES problem with some changes. Some

continuity conditions on the data ensures the solution of the problem. (The stability of the problem is also studied and it was found that  $\|a - a_N\| = O(N^{-4})$  for  $0 \leq t \leq T$ .)

CANNON (1967) [24] solves a nonlinear coefficient problem. The overdetermined data is considered to possess a continuous derivative with respect to arc length in order to solve the problem. In (CANNON, 1973) [28] the authors determined the positive function  $\{a(u), b(u), u(x,t)\}$  from the equation  $a(u)u_t = (b(u)u_x)_x$ . They have also shown that for a constant  $k > 0$  such that  $b(u) = k a(u)$  there exists a unique solution provided that the additional data is specified where  $k$  is unknown. Both for semi infinite and finite intervals, using variational formulation of nonlinear inverse problem for the determination of coefficient has been solved in (CANNON-DUCHATEAU, 1980) [29]. The authors have used maximum principle to solve the problem. Earlier a related problem was studied by ISKENDROV (1975) [70] as well. G. CHAVENT and P. LEMONNIER (1974), used an optimal control theory using interior measured data. But from the point of view of applications the problem involving interior measurements cause a lot of difficulties. DUCHATEAU (1981) [41] showed that the mapping from the coefficient to overposed data is strictly monotone (isotonic) and as a consequence the solution of the inverse problem, was shown to be unique. Under appropriate conditions, the spatially varying coefficient problem can be treated as a first order hyperbolic equation in the unknown coefficient (RICHTER, 1981) [105]. If the forcing function is positive and Hölder continuous in the domain and  $u = 0$  on the boundary then they obtain that the solution for the diffusion coefficient does not need Cauchy data to obtain an a priori bound on its stability in terms of relevant properties of the domain,  $f$  and the unknown diffusion coefficient.

In (RUNDELL, 1987) [108] the author points out the demerits of the earlier methods chosen by PIERCE (1979) [100] and SUZUKI (1983) [115] using

the technique of inverse Sturm-Liouville operators developed by GELFAND and LEVITAN. Although they have shown the existence and uniqueness of the solution pair for spatially dependent coefficient problem from overdetermined data; still the problem is not completely resolved. Since Gelfand-Levitan approach is limited to Ordinary Differential Equations therefore attempts to recover  $a(x)$  (from  $u_t - \Delta u + a(x) u = 0$ ) for  $x \in \mathbb{R}^n$  with  $n > 1$  has not proved successful. RUNDELL's integral operator method is limited to one space variable. He also discusses the nature of the prescribed initial-boundary conditions to get ill or well posed ness of the parabolic equation. An example has been furnished to show that one can not in general obtain continuous dependence on the boundary data unless the class of coefficients considered is suitably restricted.

The computational algorithm for inhomogeneous quasi linear heat conduction equation is studied in (ARTYUKHIN-NENAROKOMOV, 1987) [9].

A bibliography on inverse problems for parabolic equations can be found in CANNON's book (1984) [26].

## 1.8 SOURCE PROBLEM :

In (SOLOVEV, 1989) [110] the author determines the source term for a parabolic equation where the source  $F(x,t) = f(x) h(x,t) + g(x,t)$  is given in this fashion. The only unknown quantity here is  $f(x)$ . The overdetermined data is given on the upper base of the cylinder. A good survey is presented connected with this problem. It is found that the problem is Fredholm solvable and with some sufficient conditions the solution is unique. The solution is determined in the function space  $[u \in H^{2+\alpha, 1+\alpha/2}(\bar{Q}_T)$  and  $f \in H^\alpha(\bar{\Omega})]$ .

In (DUCHATEAU-RUNDELL, 1985) [42], the authors seek to determine an unknown source term in a reaction-diffusion equation. The analysis is based on the observation that the over specified data depends monotonically on the

unknown source term in the equation. This monotonicity is used to establish the unicity result for the inverse problem. Methods which apply to a general class of problems are very few. Monotonicity methods, however, apply to a variety of inverse problems involving partial differential equations of parabolic or elliptic type. The results presented there do not rely on special representations and hence valid for  $n$ -dimensional regions and for general parabolic operators, although for the sake of simplicity it has been restricted to  $n$ -dimensional heat operator. If the unknown function is restricted to analytic class then uniqueness for all choices of initial data follows.

CANNON (1968) [25] proposed to determine the spatial source in an  $n$ -dimensional setting considering the eigen values and normalized eigen functions of difference of two solutions of the source problem. He establishes the uniqueness, and the procedure itself gives the essence of a numerical procedure. In order to specify the class of function where the continuous dependence holds, it is necessary that the source should possess a first derivative.

ISAKOV (1991) [69] has used the monotonicity condition to study the inverse problems when the right side or a coefficient of a parabolic equation is unknown. He proves sharp estimates of solutions to the inverse problem in Hölder spaces and reduces this problem to a fredholm equation. One important counter example emphasizes the importance of the monotonicity condition for the uniqueness theorem.

In (ANGER, 1990) [7] and (PILANT-RUNDELL, (1987, 1988)) [103], [104] the authors have solved the inverse source problem for a heat conduction equation by reformulating it as a fixed point problem. This motivated us to look at other problems from the point of view.

## 1.9 PREVIEW OF THE THESIS :

In chapter two a system of two reaction-diffusion equations are considered. As in standard forms of reaction diffusion equations, the source term is assumed to be the product of two functions. The function involving the reaction is to be determined from the available data measured at the boundary. The problem is reformulated as a fixed point problem. The basic results are presenting an iterative algorithm for construction of  $f$  and showing the existence of such an  $f$  under certain conditions. In chapter three, the uniqueness of the inverse source problem and the convergence of the iterative procedure are established.

In chapter four we have considered a nonlinear source function problem for a heat conduction equation in a spatially finite region via the overdetermined data at the boundaries. The problem is again reformulated as a fixed point problem on a function space. Existence of the source term is established in chapter four and in chapter five, uniqueness as well as the convergence of iterative method has been proved.



## CHAPTER II

### EXISTENCE OF A SOLUTION OF THE SOURCE PROBLEM

#### IN A REACTION-DIFFUSION SYSTEM

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#### 2.1 INTRODUCTION :

The Reaction-Diffusion equations are the simultaneous equations arising in almost all chemical reactions. The basic equations governing these reactions are derived from the conservation of mass and heat. Chemical reactions are classified into homogeneous and heterogeneous processes, depending on whether they take place in a single phase or at an interface. In both cases, the rate of reaction is a function of the temperature and of the concentrations of the reactants (FIFE, FRANK-KANETSKII, STACKGOLD) [47], [48], [112].

Often, in practice, the gradients corresponding to both concentration and temperature are present in the same system and heat transfer and diffusion occur simultaneously. The process then becomes more complex and entirely new phenomena, known as thermal diffusion and diffusion thermo effect, arise. This is essentially due to the fact that the heat flow depends not only on the temperature gradient but also on the concentration gradient and the diffusion flow depends not only on the concentration but also on the temperature gradient.

The rest of the chapter consists of six sections. The inverse problem is formulated in section 2. Some basic notations and assumptions are stated in section 3. In section 4 an iteration scheme is developed and in section 5 it is shown that the original problem is equivalent to a fixed point problem. Existence of the fixed point is shown in section 6. The chapter ends with a brief conclusion in section 7.

## 2.2 FORMULATION OF INVERSE PROBLEM :

In this chapter we consider the inverse problem dealing with the determination of source term in a system of two reaction diffusion equations where the source term is assumed to be the product of two functions  $f(u) g(v)$  where  $g$  is assumed to be known. Our basic problem consists of an initial boundary value problem for a system of two coupled equations given in a semi-infinite domain.

$$\partial_t u - \partial_{xx} u = f(u) g(v) \quad , \quad x > 0, t > 0 ; \quad (2.2.1a)$$

$$u(x,0) = u_0(x) \quad , \quad 0 \leq x < \infty ; \quad (2.2.1b)$$

$$u_x(0,t) = g_1(t) \quad , \quad t > 0 ; \quad (2.2.1c)$$

and

$$\partial_t v - \partial_{xx} v = - f(u) g(v) \quad , \quad x > 0, t > 0 ; \quad (2.2.2a)$$

$$v(x,0) = v_0(x) \quad , \quad 0 \leq x < \infty ; \quad (2.2.2b)$$

$$v_x(0,t) = g_2(t) \quad , \quad t > 0 ; \quad (2.2.2c)$$

where  $f$  is an unknown function of  $u$  and  $g$  is a known function of  $v$ . Since the data available is insufficient so as to solve the inverse problem (that is to determine the structure of  $f$ ) one needs some additional informations such as the values of the solution at the boundary. Assume that the measured value at  $x = 0$  is given to be

$$u(0,t) = \theta(t) , \quad t > 0 ; \quad (2.2.3)$$

Now the overdetermined data is imposed on equation (2.2.1a) at  $x = 0$ . Hence it will become

$$\partial_t u(0,t) - \partial_{xx} u(0,t) = f(u(0,t)) g(v(0,t))$$

$$\text{that is,} \quad \theta'(t) - u_{xx}(0,t) = f(\theta(t)) g(v(0,t)) \quad (2.2.4)$$

## 2.3 NOTATIONS AND ASSUMPTIONS :

We shall use the following notations throughout the rest of the work.

$$\Omega = \left\{ (x,t) \mid 0 < x < \infty, 0 < t < T \right\} = \mathbb{R}^+ \times (0,T) \quad (2.3.1)$$

The following notations for norms and semi-norms are defined for functions defined on  $\Omega$  or on appropriate domains :

$$\|u\|_{\infty} = \sup \left\{ |u(x,t)| : (x,t) \in \Omega \right\} \quad (2.3.2)$$

$$|u(\cdot, t)|_{\alpha} = \sup_{\xi \neq \eta} \left\{ \frac{|u(\xi, t) - u(\eta, t)|}{|\xi - \eta|^{\alpha}} : \xi, \eta \in \mathbb{R}^+ \right\} \quad (2.3.3)$$

$$\|f\|_{\infty} = \sup \left\{ |f(\xi)| : \xi \in \text{dom } (f) \right\} \quad (2.3.4)$$

$$|f|_{\alpha} = \sup_{\xi \neq \eta} \left\{ \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|^{\alpha}} , \xi, \eta \in \text{dom } (f) \right\} \quad (2.3.5)$$

$$\|f\|_{\alpha} = \|f\|_{\infty} + |f|_{\alpha} \quad (2.3.6)$$

$C$  and  $C^{\alpha}$  denote respectively the space of the continuous functions and the space of Lipschitz continuous functions of order  $\alpha$  on appropriate domains.

The role of the time variable is one of the complicating factors, so the usual semi-norm for solutions of parabolic equations is not considered here. The following assumptions are made on the data.

$$A1: \quad u_0 \quad \text{and} \quad v_0 \in C^{2+\alpha} [0, \infty) \quad \text{and} \quad g_1 = g_2 = 0.$$

$$A2: \quad \text{The over posed data } \theta \text{ is a monotone function, } \theta' \in C^\alpha [0, \infty) \\ \text{and } \inf_{t \geq 0} |\theta'(t)| \geq \delta > 0 \text{ for some } \delta.$$

$$A3: \quad f, g \in B_E \equiv \left\{ \| \cdot \|_\alpha \leq E \right\} \text{ for some constant } E.$$

## 2.4 ITERATION SCHEME :

In order to solve the inverse problem (2.2.1) and (2.2.2) numerically, an iteration scheme is developed. If  $f(u)$  were known then (2.2.1)-(2.2.2) would define a well posed problem for the functions  $(u, v)$ . MORGAN, HOLLIS, MARTIN, PIERRE [92,61,60,62] and others have studied the direct problem dealing with global existence and the well-posedness of the reaction diffusion systems under various boundary conditions. If one assumes  $f = f^{(0)}$  satisfying desirable properties, we shall have the solutions  $u = u^{(0)} = u(x, t; f^{(0)}, g)$  and  $v = v^{(0)} = v(x, t; f^{(0)}, g)$ .

However,  $u^{(0)}$ , so obtained, will not, in general satisfy the superposed B. C. (2.2.3). So the function  $f^{(0)}$  is updated with the help of the equation (2.2.4) to a new function  $f^{(1)}$  defined as follows.

$$f^{(1)}(\theta(t)) \quad g(v^{(0)}(0, t)) = \theta'(t) - u_{xx}(0, t; f^{(0)}, g) \quad (2.4.1)$$

The basic idea behind the updating scheme is the following. If "f" is the proper choice of function, then

$$u_t(0,t) - u_{xx}(0,t) = f(u(0,t)) g(v(0,t))$$

will be satisfied. Since  $u(0,t) = \theta(t)$ ,

$$f(\theta(t)) g(v(0,t)) = \theta'(t) - u_{xx}(0,t)$$

The above expression motivates the updating procedure.

This procedure is continued and the general iteration scheme is given by

$$f^{(k+1)}(\theta(t)) g(v(0,t;f^{(k)},g)) = \theta'(t) - u_{xx}(0,t;f^{(k)},g) \quad (2.4.2)$$

for  $k = 0, 1, 2, \dots$

In view of the equation (2.2.1), we have the following equivalent form for the iteration :

$$\begin{aligned} & f^{(k+1)}(\theta(t)) g(v(0,t;f^{(k)},g)) \\ &= \theta'(t) - u_t^{(k)}(0,t;f^{(k)},g) + f^{(k)}(u^{(k)}(0,t)) g(v^{(k)}(0,t)) \\ &= f^{(k)}(\theta(t)) g(v^{(k)}(0,t)) \\ &+ \left\{ \left[ f^{(k)}(u^{(k)}(0,t)) - f^{(k)}(\theta(t)) \right] g(v^{(k)}(0,t)) + \theta'(t) - u_t^{(k)}(0,t;f^{(k)},g) \right\} \\ &= f^{(k)}(\theta(t)) g(v^{(k)}(0,t)) + F [\theta(t) - u^{(k)}(0,t)] \end{aligned} \quad (2.4.3)$$

Where  $F$  is a nonlinear map.

## 2.5 EQUIVALENCE RELATION :

$$\text{Let} \quad T_\theta[f](t) \equiv \frac{\theta'(t) - u_{xx}(0,t;f,g)}{g(v(0,t;f,g))} \quad (2.5.1)$$

denote a nonlinear map in the function space  $B_E$ . Then "f" is called a  $\theta$ -fixed point of  $T_\theta$ , if  $f(\theta(t)) = T_\theta[f](t)$ .

In view of (2.5.1) the iteration scheme can be written as

$$f^{(k+1)}(\theta(t)) = T_\theta[f^{(k)}](t) \quad (2.5.2)$$

The following lemma establishes the equivalence of the original inverse problem and determination of a fixed point of the map  $T_\theta$ .

**LEMMA 1 :**

Let  $f$  satisfies the Lipschitz condition. Then the pair  $\{u, f\}$  is a solution of the Reaction-Diffusion equations if and only if  $f$  is a  $\theta$ -fixed point of  $T_\theta$ .

**PROOF :**

Let us assume that  $\{u, f\}$  is a solution of R-D equations. Our aim is to show that  $f$  is a fixed point of  $T_\theta$ .

Since  $\{u, f\}$  is a solution, it satisfies the over posed data,

$$u(0, t) = \theta(t)$$

Therefore,

$$f(\theta(t)) = f(u(0, t))$$

$$\begin{aligned} &= \frac{u_t(0, t) - u_{xx}(0, t; f, g)}{g(v(0, t; f, g))} \\ &= \frac{\theta'(t) - u_{xx}(0, t; f, g)}{g(v(0, t; f, g))} \\ &= T_\theta[f](t) \end{aligned}$$

Hence  $f(\theta(t)) = T_\theta[f](t)$ . This implies that  $f$  is a  $\theta$ -fixed point of  $T_\theta$ .

## CONVERSELY:

Suppose  $f$  satisfies the Lipschitz condition and  $f$  is a  $\theta$ -fixed point of  $T_\theta$ . It is sufficient to show  $u(0,t;f,g) = \theta(t)$ .

Indeed, from (2.2.1), for  $x = 0$ , we get

$$f(u(0,t)) \ g(v(0,t)) = u_t(0,t) - u_{xx}(0,t;f,g)$$

$$\Leftrightarrow u_{xx}(0,t;f,g) = u_t(0,t) - f(u(0,t)) \ g(v(0,t))$$

Since  $f$  is a  $\theta$ -fixed point of  $T_\theta$ ,

$$f(\theta(t)) = T_\theta[f](t) = \frac{\theta'(t) - u_{xx}(0,t;f,g)}{g(v(0,t))}$$

$$\begin{aligned} \Rightarrow f(\theta(t)) \ g(v(0,t)) &= \theta'(t) - u_{xx}(0,t;f,g) \\ &= \theta'(t) - u_t(0,t) + f(u(0,t)) \ g(v(0,t)) \end{aligned}$$

$$\Rightarrow \theta'(t) - u_t(0,t) = \left[ f(\theta(t)) - f(u(0,t)) \right] g(v(0,t)) \quad (2.5.3)$$

$$\text{Let } \alpha(t) = \theta(t) - u(0,t) \quad (2.5.4)$$

If  $f$  has Lipschitz constant  $C$  and

$$|g|_\infty = \sup_v |g(v(x,t))|$$

Then (2.5.3) can be estimated as

$$\begin{aligned} |\alpha'(t)| &\leq |[f(\theta(t)) - f(u(0,t))]| |g(v(0,t))| \\ &\leq C |g|_\infty |\theta(t) - u(0,t)| \\ &\leq C_1 |\alpha(t)| \end{aligned} \quad (2.5.5)$$

where  $C_1 = C |g|_\infty$

We claim  $\alpha(t) \equiv 0$  for all  $t$ .

Note that the compatibility condition ensures that  $\theta(t)$  must agree with the initial condition  $u_0(x)$  at  $x = 0$ . So for  $t = 0$ ,  $\alpha(0) = \theta(0) - u(0,0) = 0$ .

Now suppose  $\alpha(t) \not\equiv 0$ . Represent  $\alpha(t)$  as

$$\alpha(t) = \int_0^t \alpha'(\tau) d\tau \quad (2.5.6)$$

Then

$$|\alpha(t)| \leq \int_0^t |\alpha'(\tau)| d\tau \leq C_1 \int_0^t |\alpha(\tau)| d\tau$$

Thus by Gronwall's lemma  $\alpha(t) \equiv 0$ , is a contradiction. So our claim implies,

$$u(0,t;f,g) = \theta(t)$$

In other words  $\{u,f\}$  is a solution of the R-D equations. This completes the proof of the Lemma. ■

## 2.6 EXISTENCE OF FIXED POINT :

The main aim of the chapter is to show the existence of a fixed point of the R-D source problem. Therefore, in this section it has shown that for  $0 < \alpha < 1$ ,  $T_\theta : C^\alpha \longrightarrow C^\alpha$  and from SCHAUDER's Theorem one can get the required result ( see FRIEDMANN [49]).

Before proceeding to prove the main result of this section we perform some preliminary preparations for subsequent use. Consider the comparison function  $\psi$ , which satisfies the differential equations,

$$\begin{aligned} \partial_t \psi(x,t) - \partial_{xx} \psi(x,t) &= 0 \quad , \quad x > 0, t > 0 ; \\ \psi(x,0) &= u_0(x) \quad , \quad x > 0 ; \\ \psi_x(0,t) &= g_1(t) \quad , \quad t > 0 ; \end{aligned} \quad (2.6.1)$$

If one writes  $r = u - \psi$ , then  $r$  will satisfy

$$\begin{aligned}
\partial_t r - \partial_{xx} r &= f(u) g(v), \quad x > 0, \quad t > 0; \\
r(x, 0) &= 0, \quad x > 0; \\
r_x(0, t) &= 0, \quad t > 0;
\end{aligned} \tag{2.6.2}$$

Hence the solution (2.6.2) can be represented as (see CANNON [26])

$$r = u - \psi = \int_0^t \int_0^\infty K(x, y, t-\tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \tag{2.6.3}$$

where  $K$  is the Neumann function for the semi-infinite domain of the homogeneous equations (2.6.1). The expression for  $K$  is as follows (STAKGOLD [111]).

$$K = K(x, y, t) = \frac{1}{2\sqrt{\pi}} \left[ \frac{e^{-(x-y)^2/4t}}{\sqrt{t}} + \frac{e^{-(x+y)^2/4t}}{\sqrt{t}} \right] \tag{2.6.4}$$

This Neumann function satisfies

$$\int_0^\infty K_x(x, y, t-\tau) dy = 0 = \int_0^\infty K_{xx}(x, y, t-\tau) dy \tag{2.6.5}$$

Hence for any function  $h \in C^\alpha$ ,  $0 < \alpha < 1$ , the following is true

$$\begin{aligned}
\int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) h(x, \tau) dy d\tau &= \int_0^t h(x, \tau) \left[ \int_0^\infty K_{xx}(x, y, t-\tau) dy \right] d\tau \\
&= 0
\end{aligned} \tag{2.6.6}$$

Therefore, we have the following identity property.

$$\int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) h(y, \tau) dy d\tau = \int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) [h(y, \tau) - h(x, \tau)] dy d\tau \tag{2.6.7}$$

During the course of the proof of the main result, the above identity is repeatedly used and this is the basic tool for proving the theorem. Differentiating (2.6.3) twice with respect to  $x$ , we get

$$r_{xx} = u_{xx} - \psi_{xx} = \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau \quad (2.6.8)$$

$$\Rightarrow u_{xx}(x,t;f,g) = \psi_{xx}(x,t) + \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau \quad (2.6.9)$$

Evaluating at  $x = 0$ , we get the following relation

$$\begin{aligned} u_{xx}(0,t;f,g) &= \psi_{xx}(0,t) + K[f(u) g(v)] \\ &= \psi_t + K[f(u) g(v)] \end{aligned} \quad (2.6.10)$$

where

$$K[f(u) g(v)] = \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau \quad (2.6.11)$$

In view of the relation (2.2.4) at  $x = 0$ , (2.6.10) reduces to

$$\begin{aligned} T_\theta[f](t) &= f(\theta(t)) \\ &= \frac{\theta'(t) - u_{xx}(0,t;f,g)}{g(v(0,t;f,g))} \\ &= \frac{\theta'(t) - \psi_t(0,t) + \psi_{xx}(0,t) - u_{xx}(0,t;f,g)}{g(v(0,t;f,g))} \\ &= \frac{\theta'(t) - \psi_t(0,t) - K[f(u) g(v)]}{g(v(0,t;f,g))} \end{aligned} \quad (2.6.12)$$

So

$$f(\theta(t)) = T_\theta[f](t)$$

$$= \frac{\theta'(t) - \psi_t(0,t) - \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau}{g(v(0,t;f,g))}$$

Hence,

$$f(\theta(t)) g(v(0,t;f,g)) = \theta'(t) - \psi_t(0,t) - \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau \quad (2.6.13)$$

We now state the following Theorem.

**THEOREM 1 :**

If the assumptions (A1)-(A3) are true, then for sufficiently small  $T$ ,  $T_\theta$  maps  $C^\alpha [0,T]$  into itself.

**PROOF :**

Given a function  $f \in C^\alpha$ , define the function  $\tilde{f}$  by

$$\tilde{f}(\theta(t)) = T_\theta[f](t).$$

For a monotone function " $\theta$ ",  $\tilde{f}$  is a single valued function.

To show  $T_\theta : C^\alpha \longrightarrow C^\alpha$ , it is enough to show  $\|\tilde{f}\|_\alpha$  is bounded by  $\|f\|_\alpha$  for  $0 < \alpha < 1$ . Equation (2.6.13) can be written in an equivalent form, by applying the identity property (2.6.7) at  $x = 0$ . Omitting the dependence of  $v$  on  $f, g$  in the notation, the following equation is obtained.

$$\begin{aligned} \tilde{f}(\theta(t)) g(v(0,t)) &= \theta'(t) - \psi_t(0,t) \\ &- \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) \left[ f(u(y,\tau)) g(v(y,\tau)) - f(u(0,\tau)) g(v(0,\tau)) \right] dy d\tau \end{aligned} \quad (2.6.14)$$

By adding and subtracting some suitable terms, the bracketed expression appearing within the integral sign becomes

$$\begin{aligned}
& f(u(y,\tau)) g(v(y,\tau)) - f(u(0,\tau)) g(v(0,\tau)) \\
&= [f(u(y,\tau)) - f(u(0,\tau))] g(v(y,\tau)) \\
&+ f(u(0,\tau)) [g(v(y,\tau)) - g(v(0,\tau))] \quad (2.6.15)
\end{aligned}$$

Assuming  $f, g \in C^\alpha$ , (2.6.14) can be estimated as

$$\begin{aligned}
|\tilde{f}(\theta(t))| \leq & \frac{1}{|g(v(0,t))|} \left[ |\theta'(t) - \psi_t(0,t)| \right. \\
& \left. + \int_0^t \int_0^\infty \left[ |K_{xx}(0,y,t-\tau)| |f(u(y,\tau)) g(v(y,\tau)) - f(u(0,\tau)) g(v(0,\tau))| \right] dy d\tau \right] \quad (2.6.16)
\end{aligned}$$

Consider the second term of (2.6.16), which is denoted by  $I_1$  and substitute equation (2.6.15) into  $I_1$ . Since  $f$  and  $g$  are in  $C^\alpha[0,\infty)$ , then it becomes

$$\begin{aligned}
I_1 \leq & \int_0^t \int_0^\infty |K_{xx}(0,y,t-\tau)| \left[ |f(u(y,\tau)) - f(u(0,\tau))| |g(v(y,\tau))| \right. \\
& \left. + |f(u(0,\tau))| |g(v(y,\tau)) - g(v(0,\tau))| \right] dy d\tau \\
\leq & \int_0^t \int_0^\infty |K_{xx}(0,y,t-\tau)| \left[ |g|_\infty |f|_\alpha |u(y,\tau) - u(0,\tau)|^\alpha \right. \\
& \left. + |f|_\infty |g|_\alpha |v(y,\tau) - v(0,\tau)|^\alpha \right] dy d\tau
\end{aligned}$$

Now, expanding  $u$  and  $v$  at the point  $y = 0$  and assuming  $u_x(0,t) = g_1(t) = 0$  and  $v_x(0,t) = g_2(t) = 0$ , we get

$$\begin{aligned}
I_1 \leq & \int_0^t \int_0^\infty |K_{xx}(0,y,t-\tau)| \left[ |f|_\alpha |g|_\infty (|y|^{2\alpha}/2^\alpha) (|u_{xx}|_\infty)^\alpha \right. \\
& \left. + |f|_\infty |g|_\alpha (|y|^{2\alpha}/2^\alpha) (|v_{xx}|_\infty)^\alpha \right] dy d\tau
\end{aligned}$$

Since (see Appendix A , (A6))

$$\int_0^t \int_0^\infty |K_{xx}(0, y, t-\tau)| |y|^\alpha dy d\tau \leq \frac{1}{2\alpha} C_3(\alpha) t^{\alpha/2}$$

one gets,

$$I_1 \leq 2^{-(\alpha+2)} C_3(2\alpha) (t^\alpha/\alpha) \left[ |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha + |g|_\alpha |f|_\infty (|v_{xx}|_\infty)^\alpha \right] \quad (2.6.17)$$

Hence, the sup-norm of  $\tilde{f}$  can be given by

$$|\tilde{f}|_\infty \leq (\inf |g|)^{-1} \left[ |\theta' - \psi_t|_\infty + 2^{-(\alpha+2)} C_3(2\alpha) (t^\alpha/\alpha) \times \right. \\ \left. \left[ |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha + |g|_\alpha |f|_\infty (|v_{xx}|_\infty)^\alpha \right] \right] \quad (2.6.18)$$

In order to compute the  $\alpha$ -norm of  $\tilde{f}$ , which is defined as  $\|\tilde{f}\|_\alpha = |\tilde{f}|_\infty + |\tilde{f}|_\alpha$ , we now set out to compute the  $\alpha$ -semi norm of  $\tilde{f}$ . Without any loss of generality it is assumed that  $t_1 > t_2$ . We calculate the value of  $\tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2))$  first.

$$\tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2)) =$$

$$\frac{\theta'(t_1) - \psi_t(0, t_1) - \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau}{g(v(0, t_1; f, g))}$$

$$- \frac{\theta'(t_2) - \psi_t(0, t_2) - \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) \left[ f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau}{g(v(0, t_2; f, g))}$$

Therefore,

$$\begin{aligned}
\tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2)) &= \frac{1}{g(v(0, t_1)) g(v(0, t_2))} \times \\
&\left[ g(v(0, t_2)) \left\{ \theta'(t_1) - \psi_t(0, t_1) \right\} - g(v(0, t_1)) \left\{ \theta'(t_2) - \psi_t(0, t_2) \right\} \right. \\
&- g(v(0, t_2)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&\left. + g(v(0, t_1)) \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right] \quad (2.6.19)
\end{aligned}$$

At this point, only the following terms are considered.

$$\begin{aligned}
&g(v(0, t_2)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&- g(v(0, t_1)) \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau
\end{aligned}$$

Adding and subtracting

$$g(v(0, t_1)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau$$

from the above expression, one gets

$$\begin{aligned}
&+ g(v(0, t_2)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&- g(v(0, t_1)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&+ g(v(0, t_1)) \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&- g(v(0, t_1)) \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau
\end{aligned}$$

$$\begin{aligned}
&= \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&+ g(v(0, t_1)) \left[ \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right. \\
&\left. - \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right] \quad (2.6.20)
\end{aligned}$$

Similarly, the other two terms of (2.6.19) can be written as

$$\begin{aligned}
&g(v(0, t_2)) \left\{ \theta'(t_1) - \psi_t(0, t_1) \right\} - g(v(0, t_1)) \left\{ \theta'(t_2) - \psi_t(0, t_2) \right\} \\
&= \left\{ \theta'(t_1) - \psi_t(0, t_1) \right\} \times \left\{ g(v(0, t_2)) - g(v(0, t_1)) \right\} \\
&+ g(v(0, t_1)) \left\{ \theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2) \right\} \quad (2.6.21)
\end{aligned}$$

Combining (2.6.20) and (2.6.21), (2.6.19) can be represented as

$$\begin{aligned}
&\tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2)) \\
&= \frac{1}{g(v(0, t_1)) g(v(0, t_2))} \left[ \left\{ g(v(0, t_2)) - g(v(0, t_1)) \right\} \right. \\
&\times \left\{ \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right\} \\
&+ \left\{ g(v(0, t_2)) - g(v(0, t_1)) \right\} \times \left\{ \theta'(t_1) - \psi_t(0, t_1) \right\} \\
&+ g(v(0, t_1)) \left\{ \theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2) \right\} \\
&+ g(v(0, t_1)) \left[ \left\{ \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right\} \right.
\end{aligned}$$

$$- \left\{ \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \right\} \Big] \Big] \quad (2.6.22)$$

Now all the terms of (2.6.22) are simplified one after the other.

Consider  $I_2$  first, where

$$\begin{aligned} I_2 &= \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\ &- \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \end{aligned}$$

Since  $t_2 < t_1$ , therefore,

$$\begin{aligned} I_2 &= \int_0^{t_2} \int_0^\infty \left\{ \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau \\ &+ \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \end{aligned}$$

$$I_2 = I_{2,1} + I_{2,2} \quad (2.6.23)$$

These two terms are separately estimated. Using the mean value theorem,

$I_{2,1}$  can be written as

$$\begin{aligned} I_{2,1} &= \int_0^{t_2} \int_0^\infty \left\{ \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau \\ &= \int_0^{t_2} \int_0^\infty \int_{t_2}^{t_1} \left\{ K_{xxt}(0, y, l - \tau) \left[ f(u(y, \tau)) g(v(y, \tau)) \right. \right. \\ &\quad \left. \left. - f(u(0, \tau)) g(v(0, \tau)) \right] \right\} dl dy d\tau \end{aligned}$$

$$= \int_0^{t_2} \int_0^\infty \int_{t_2}^{t_1} \left\{ K_{\text{xtt}}(0, y, 1-\tau) \left[ g(v(y, \tau)) \left\{ f(u(y, \tau)) - f(u(0, \tau)) \right\} \right. \right. \\ \left. \left. + f(u(0, \tau)) \left\{ g(v(y, \tau)) - g(v(0, \tau)) \right\} \right] \right\} dl \, dy \, d\tau$$

Taking the estimate on both the sides

$$|I_{2,1}| \leq \int_0^{t_2} \int_0^\infty \int_{t_2}^{t_1} \left\{ |K_{\text{xtt}}(0, y, 1-\tau)| \left[ |g(v(y, \tau))| |f(u(y, \tau)) - f(u(0, \tau))| \right. \right. \\ \left. \left. + |f(u(0, \tau))| |g(v(y, \tau)) - g(v(0, \tau))| \right] \right\} dl \, dy \, d\tau \\ \leq \int_0^{t_2} \int_0^\infty \int_{t_2}^{t_1} \left\{ |K_{\text{xtt}}(0, y, 1-\tau)| \left[ |f|_\alpha |g|_\infty (|u_{\text{xx}}|_\infty)^\alpha (|y|^{2\alpha}/2^\alpha) \right. \right. \\ \left. \left. + |f|_\infty |g|_\alpha (|v_{\text{xx}}|_\infty)^\alpha (|y|^{2\alpha}/2^\alpha) \right] \right\} dl \, dy \, d\tau \\ \leq 2^{-\alpha} \left[ |f|_\alpha |g|_\infty (|u_{\text{xx}}|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_{\text{xx}}|_\infty)^\alpha \right] \times \\ \int_{t_2}^{t_1} \int_0^{t_2} \int_0^\infty |K_{\text{xtt}}(0, y, 1-\tau)| |y|^{2\alpha} dy \, d\tau \, dl$$

Making use of (A7) from Appendix A one can derive

$$\leq 2^{-\alpha} \left[ |f|_\alpha |g|_\infty (|u_{\text{xx}}|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_{\text{xx}}|_\infty)^\alpha \right] C_4(2\alpha) \int_{t_2}^{t_1} \int_0^{t_2} |1-\tau|^{\alpha-2} d\tau \, dl$$

Therefore,  $I_{2,1}$  can be weighted as

$$|I_{2,1}| \leq \frac{2^{-\alpha} C_4(2\alpha)}{\alpha(1-\alpha)} \left[ |f|_\infty |g|_\alpha (|v_{\text{xx}}|_\infty)^\alpha + |f|_\alpha |g|_\infty (|u_{\text{xx}}|_\infty)^\alpha \right] |t_1 - t_2|^\alpha$$

(2.6.24)

Now  $I_{2,2}$ , the other term of equation (2.6.23) is estimated as follows.

$$\begin{aligned}
 I_{2,2} &= \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
 &= \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ f(u(y, \tau)) g(v(y, \tau)) - f(u(0, \tau)) g(v(0, \tau)) \right] dy d\tau \\
 &= \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ g(v(y, \tau)) \left\{ f(u(y, \tau)) - f(u(0, \tau)) \right\} \right. \\
 &\quad \left. + f(u(0, \tau)) \left\{ g(v(y, \tau)) - g(v(0, \tau)) \right\} \right] dy d\tau
 \end{aligned}$$

So taking the modulus on both the sides, we have

$$\begin{aligned}
 |I_{2,2}| &\leq \int_{t_2}^{t_1} \int_0^\infty |K_{xx}(0, y, t_1 - \tau)| \left[ |g(v(y, \tau))| |f(u(y, \tau)) - f(u(0, \tau))| \right. \\
 &\quad \left. + |f(u(0, \tau))| |g(v(y, \tau)) - g(v(0, \tau))| \right] dy d\tau \\
 &\leq \int_{t_2}^{t_1} \int_0^\infty |K_{xx}(0, y, t_1 - \tau)| \left[ |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha (|y|^{2\alpha}/2^\alpha) \right. \\
 &\quad \left. + |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha (|y|^{2\alpha}/2^\alpha) \right] dy d\tau \\
 &\leq \left\{ |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha \right\} \\
 &\quad \times \left\{ \int_{t_2}^{t_1} \int_0^\infty |K_{xx}(0, y, t_1 - \tau)| (|y|^{2\alpha}/2^\alpha) dy d\tau \right\} \\
 &\leq 2^{-(\alpha+2)} \frac{C_3(2\alpha)}{\alpha} \left[ |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha \right] |t_1 - t_2|^\alpha
 \end{aligned}$$

Therefore, (2.6.23) becomes,

$$\begin{aligned}
 |I_2| &\leq |I_{2,1}| + |I_{2,2}| \\
 &\leq \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |f|_{\infty} |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} + |f|_{\alpha} |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} \right] |t_1 - t_2|^{\alpha}
 \end{aligned}
 \tag{2.6.26}$$

Where,

$$C(\alpha) = 2^{-\alpha} C_4(2\alpha) + 2^{-(\alpha+2)} (1-\alpha) C_3(2\alpha)$$

Now represent the first term of (2.6.22) by  $I_3$

$$\begin{aligned}
 I_3 &= \left\{ g(v(0, t_2)) - g(v(0, t_1)) \right\} \times \\
 &\quad \int_0^{t_1} \int_0^{\infty} K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
 \Rightarrow |I_3| &\leq \left\{ |g(v(0, t_2)) - g(v(0, t_1))| \right\} \\
 &\quad \times \int_0^{t_1} \int_0^{\infty} |K_{xx}(0, y, t_1 - \tau)| \left\{ |f(u(y, \tau)) g(v(y, \tau)) - f(u(0, \tau)) g(v(0, \tau))| dy d\tau \right\}
 \end{aligned}$$

One can expand  $v(0, t_1)$  considering the centre at  $t_2$  and the radius as  $(t_1 - t_2)$ . since  $g \in C^{\alpha}$  we have from (A6)

$$\begin{aligned}
 |I_3| &\leq 2^{-(\alpha+2)} |g|_{\alpha} |v(0, t_2) - v(0, t_1)|^{\alpha} \frac{C_3(2\alpha)}{\alpha} t_1^{\alpha} \\
 &\quad \left[ |g|_{\alpha} |f|_{\infty} (|v_{xx}|_{\infty})^{\alpha} + |g|_{\infty} |f|_{\alpha} (|u_{xx}|_{\infty})^{\alpha} \right] \\
 &\leq 2^{-(\alpha+2)} |g|_{\alpha} |v_t|^{\alpha} |t_1 - t_2|^{\alpha} \frac{C_3(2\alpha)}{\alpha} t_1^{\alpha} \\
 &\quad \left[ |g|_{\alpha} |f|_{\infty} (|v_{xx}|_{\infty})^{\alpha} + |g|_{\infty} |f|_{\alpha} (|u_{xx}|_{\infty})^{\alpha} \right]
 \end{aligned}
 \tag{2.6.27}$$

Denoting

$$I_4 = \left\{ g(v(0, t_2)) - g(v(0, t_1)) \right\} \left\{ \theta'(t_1) - \psi_t(0, t_1) \right\}$$

the second term of the equation (2.6.22) is estimated to give

$$|I_4| \leq |g|_\alpha |t_1 - t_2|^\alpha (|v_t|_\infty)^\alpha |\theta' - \psi_t|_\infty \quad (2.6.28)$$

Expressing the third term of (2.6.22) as  $I_5$  and taking modulus one gets

$$|I_5| \leq |g|_\infty \left| \theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2) \right| \quad (2.6.29)$$

Now combining (2.6.26), (2.6.27), (2.6.28) and (2.6.29), we obtain the following estimate from (2.6.22) :

$$\begin{aligned} & | \tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2)) | \\ & \leq \frac{1}{|g(v(0, t_1))| |g(v(0, t_2))|} \left[ |g|_\infty |\theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2)| \right. \\ & \quad + |g|_\alpha |t_1 - t_2|^\alpha (|v_t|_\infty)^\alpha |\theta' - \psi|_\infty \\ & \quad + 2^{-(\alpha+2)} |g|_\alpha |t_1 - t_2|^\alpha (|v_t|_\infty)^\alpha t_1^\alpha \frac{C_3(2\alpha)}{\alpha} \\ & \quad \times \left\{ |g|_\alpha |f|_\infty (|v_{xx}|_\infty)^\alpha + |g|_\infty |f|_\alpha (|u_{xx}|_\infty)^\alpha \right\} \\ & \quad \left. + \frac{C(\alpha)}{\alpha(1-\alpha)} |g|_\infty |t_1 - t_2|^\alpha \left\{ |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha + |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha \right\} \right] \end{aligned} \quad (2.6.30)$$

So the  $\alpha$ -semi norm of  $f$  is obtained by dividing both the sides of (2.6.30) by  $|\theta(t_1) - \theta(t_2)|^\alpha$  and taking the supremum.

Thus,

$$\begin{aligned}
 & \frac{|\tilde{f}(\theta(t_1)) - \tilde{f}(\theta(t_2))|}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 & \leq \left( \inf |g|^2 \right)^{-2} \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 & \times \left[ |g|_\infty \frac{|\theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2)|}{|t_1 - t_2|^\alpha} \right. \\
 & + |g|_\alpha (|v_t|_\infty)^\alpha |\theta' - \psi_t|_\infty + |g|_\alpha |t_1|^\alpha (|v_t|_\infty)^\alpha \\
 & \times \frac{C(\alpha)}{\alpha(1-\alpha)} \left\{ |g|_\alpha |f|_\infty (|v_{xx}|_\infty)^\alpha + |g|_\infty |f|_\alpha (|u_{xx}|_\infty)^\alpha \right\} \\
 & + \frac{C(\alpha)}{\alpha(1-\alpha)} |g|_\infty \left\{ |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha + |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha \right\} \Big]
 \end{aligned}$$

So this will imply,

$$\begin{aligned}
 |\tilde{f}|_\alpha & \leq \frac{(\inf |g|^2)^{-1}}{(\inf |\theta'|)^\alpha} \left[ |g|_\infty |\theta' - \psi_t|_\alpha + |g|_\alpha (|v_t|_\infty)^\alpha |\theta' - \psi_t|_\infty \right. \\
 & + \frac{C(\alpha)}{\alpha(1-\alpha)} \left( |g|_\infty + |g|_\alpha |t_1|^\alpha (|v_t|_\infty)^\alpha \right) \\
 & \times \left. \left\{ |f|_\infty |g|_\alpha (|v_{xx}|_\infty)^\alpha + |f|_\alpha |g|_\infty (|u_{xx}|_\infty)^\alpha \right\} \right] \quad (2.6.31)
 \end{aligned}$$

Here, equation (2.6.18) will be repeated for the sake of completeness.

$$|\tilde{f}|_{\infty} \leq \left( \inf |g| \right)^{-1} \left[ |\theta' - \psi_t|_{\infty} \right. \\ \left. + 2^{-\frac{(\alpha+2)}{2}} \frac{C_3(2\alpha)}{\alpha} t^{\alpha} \left\{ |f|_{\alpha} |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} + |f|_{\infty} |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} \right]$$

or,

$$|\tilde{f}|_{\infty} \leq \left( \inf |g| \right)^{-1} \left[ |\theta' - \psi_t|_{\infty} \right. \\ \left. + \frac{C(\alpha)}{\alpha(1-\alpha)} t^{\alpha} \left\{ |f|_{\alpha} |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} + |f|_{\infty} |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} \right] \quad (2.6.32)$$

Therefore, adding (2.6.31) and (2.6.32) the  $\alpha$ -norm of  $\tilde{f}$  can be found out.

$$\begin{aligned} \|\tilde{f}\|_{\alpha} &= |\tilde{f}|_{\infty} + |\tilde{f}|_{\alpha} \\ &\leq \left( \inf |g| \right)^{-1} \left[ |\theta' - \psi_t|_{\infty} \right. \\ &\quad \left. + \frac{C(\alpha)}{\alpha(1-\alpha)} t^{\alpha} \left\{ |f|_{\alpha} |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} + |f|_{\infty} |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} \right] \\ &\quad + \frac{(\inf |g|^2)^{-1}}{(\inf |\theta'|)^{\alpha}} \left[ |g|_{\infty} |\theta' - \psi_t|_{\alpha} + |g|_{\alpha} (|v_t|_{\infty})^{\alpha} |\theta' - \psi_t|_{\infty} \right. \\ &\quad \left. + \frac{C(\alpha)}{\alpha(1-\alpha)} \left( |g|_{\infty} + |t_1|^{\alpha} (|v_t|_{\infty})^{\alpha} |g|_{\alpha} \right) \right. \\ &\quad \left. \times \left\{ |f|_{\infty} |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} + |f|_{\alpha} |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} \right\} \right] \\ &\leq \left[ (\inf |g|)^{-1} + \frac{(\inf |g|^2)^{-1}}{(\inf |\theta'|)^{\alpha}} |g|_{\infty} \right. \end{aligned}$$

$$\begin{aligned}
& + \left( \inf |g|^2 \right)^{-1} |g|_{\alpha} \frac{(|v_t|_{\infty})^{\alpha}}{(\inf |\theta'|)^{\alpha}} \Big] \times \| \theta' - \psi_t \|_{\alpha} \\
& + \frac{C(\alpha)}{\alpha(1-\alpha)} \left\{ t^{\alpha} (\inf |g|)^{-1} + \frac{(\inf |g|^2)^{-1}}{(\inf |\theta'|)^{\alpha}} |g|_{\infty} \right. \\
& + |g|_{\alpha} |t_1|^{\alpha} \frac{(|v_t|)^{\alpha}}{(\inf |\theta'|)^{\alpha}} \times (\inf |g|^2)^{-1} \Big\} \\
& \times \left\{ |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} \| f \|_{\alpha} \\
& \leq M_1 \| \theta' - \psi_t \|_{\alpha} + M_2 \| f \|_{\alpha} \tag{2.6.33}
\end{aligned}$$

(2.6.33) implies that  $\| \tilde{f} \|_{\alpha}$  is bounded above by  $\| f \|_{\alpha}$ . So if  $\| f \|_{\alpha} \leq E$ , we can have  $\| \tilde{f} \|_{\alpha} \leq E$  provided  $M_1$  and  $M_2$  are small quantities. Here one needs to show,  $(|u_{xx}|_{\infty})^{\alpha}$ ,  $(|v_{xx}|_{\infty})^{\alpha}$  and  $(|v_t| / (\inf |\theta'|))^{\alpha}$  are small so that  $M_1$ ,  $M_2$  are small enough quantities.

To estimate  $u_{xx}$ , consider the relation

$$\begin{aligned}
u &= \psi + \int_0^t \int_0^{\infty} K(x, y, t-\tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
\Rightarrow u_{xx} &= \psi_{xx} + \int_0^t \int_0^{\infty} K_{xx}(x, y, t-\tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \\
&= \psi_{xx} + \int_0^t \int_0^{\infty} K_{xx}(x, y, t-\tau) \left[ f(u(y, \tau)) g(v(y, \tau)) - \right. \\
&\quad \left. f(u(x, \tau)) g(v(x, \tau)) \right] dy d\tau \\
&= \psi_{xx} + \int_0^t \int_0^{\infty} K_{xx}(x, y, t-\tau) \left[ g(v(y, \tau)) \left\{ f(u(y, \tau)) - f(u(x, \tau)) \right\} \right. \\
&\quad \left. + f(u(x, \tau)) \left\{ g(v(y, \tau)) - g(v(x, \tau)) \right\} \right] dy d\tau
\end{aligned}$$

Now taking the estimates on both the sides and expanding  $u$  and  $v$  with respect to  $x$  and from (A5), it becomes

$$\begin{aligned}
 |u_{xx} - \psi_{xx}| &\leq \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| \left\{ |g|_\infty |f|_\alpha |y-x|^\alpha (|u_x|_\infty)^\alpha \right. \\
 &\quad \left. + |f|_\infty |g|_\alpha |y-x|^\alpha (|v_x|_\infty)^\alpha \right\} dy d\tau \\
 &\leq \left[ \left\{ |g|_\infty |f|_\alpha (|u_x|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_x|_\infty)^\alpha \right\} \right. \\
 &\quad \left. \times \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |y-x|^\alpha dy d\tau \right] \\
 &\leq \left\{ |g|_\infty |f|_\alpha (|u_x|_\infty)^\alpha + |g|_\alpha |f|_\infty (|v_x|_\infty)^\alpha \right\} \frac{C_3^{(\alpha)}}{\alpha} t^{\alpha/2}
 \end{aligned} \tag{2.6.34}$$

In view of Appendix A (A3) we can similarly estimate  $|u_x|_\infty$ .

$$\begin{aligned}
 |u_x - \psi_x| &\leq \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| |f(u(y, \tau))| |g(v(y, \tau))| dy d\tau \\
 &\leq |f|_\infty |g|_\infty \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| dy d\tau \\
 &\leq C_2(0) \sqrt{t} |f|_\infty |g|_\infty
 \end{aligned} \tag{2.6.35}$$

Substituting (2.6.35) into (2.6.34), we therefore have

$$\begin{aligned}
 |u_{xx} - \psi_{xx}| &\leq \left\{ |g|_\infty |f|_\alpha (|u_x|_\infty)^\alpha + |g|_\alpha |f|_\infty (|v_x|_\infty)^\alpha \right\} \frac{C_3^{(\alpha)}}{\alpha} t^{\alpha/2} \\
 &\leq \left\{ |g|_\infty |f|_\alpha \left( |\psi_x| + C_2(0) \sqrt{t} |f|_\infty |g|_\infty \right)^\alpha + |f|_\infty |g|_\alpha (|v_x|_\infty)^\alpha \right\} \frac{C_3^{(\alpha)}}{\alpha} t^{\alpha/2}
 \end{aligned} \tag{2.6.36}$$

Let  $\phi$  be another comparison function corresponding to  $v$ , which satisfies the differential equation in a semi-infinite domain

$$\begin{aligned}\partial_t \phi - \partial_{xx} \phi &= 0, \quad x > 0, \quad t > 0; \\ \phi(x, 0) &= v_0(x), \quad x > 0; \\ \partial_x \phi(0, t) &= g_2(t), \quad t > 0;\end{aligned}\tag{2.6.37}$$

So, if  $q$  is represented by  $v - \phi$ , then  $q$  becomes

$$q = v - \phi = \int_0^t \int_0^\infty K(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau \tag{2.6.38}$$

Differentiating (2.6.38) with respect to  $x$  will give rise

$$v_x = \phi_x + \int_0^t \int_0^\infty K_x(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau \tag{2.6.39}$$

Now taking modulus on both the sides,

$$\begin{aligned}|v_x - \phi_x| &\leq |f|_\infty |g|_\infty \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| dy d\tau \\ &\leq |f|_\infty |g|_\infty C_2(0) \sqrt{t}\end{aligned}\tag{2.6.40}$$

putting (2.6.40) into (2.6.39), can be written as

$$\begin{aligned}|u_{xx} - \psi_{xx}| &\leq \left\{ |g|_\infty |f|_\alpha \left( |\psi_x| + C_2(0) \sqrt{t} |f|_\infty |g|_\infty \right)^\alpha \right. \\ &\quad \left. + |f|_\infty |g|_\alpha \left( |\phi_x| + C_2(0) \sqrt{t} |g|_\infty |f|_\infty \right)^\alpha \right\} \frac{C_3(\alpha)}{\alpha} t^{\alpha/2}\end{aligned}\tag{2.6.41}$$

This implies that

$$u_{xx} - \psi_{xx} \longrightarrow 0 \text{ like } O(t^{\alpha/2})$$

$$\text{if } \|f\|_{\alpha} \leq E \text{ and } \|g\|_{\alpha} \leq E$$

$$\Rightarrow \|u_{xx}\|_{\infty} \cong \|\psi_{xx}\|_{\infty} \cong \|u'_0\|_{\infty} \quad (2.6.42)$$

Similarly,  $v_{xx}$  can be estimated.

Since,

$$v_{xx} = \phi_{xx} + \int_0^t \int_0^{\infty} K_{xx}(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right. \\ \left. + f(u(x, \tau)) g(v(x, \tau)) \right\} dy d\tau$$

So

$$\begin{aligned} |v_{xx} - \phi_{xx}| &\leq \int_0^t \int_0^{\infty} |K_{xx}(x, y, t-\tau)| \left\{ |g(v(y, \tau))| |f(u(x, \tau)) - f(u(y, \tau))| \right. \\ &\quad \left. + |f(u(x, \tau))| |g(v(x, \tau)) - g(v(y, \tau))| \right\} dy d\tau \\ &\leq \left\{ \|g\|_{\infty} \|f\|_{\alpha} (\|u_x\|_{\infty})^{\alpha} + \|f\|_{\infty} \|g\|_{\alpha} (\|v_x\|_{\infty})^{\alpha} \right\} \\ &\quad \int_0^t \int_0^{\infty} |K_{xx}(x, y, t-\tau)| |y-x|^{\alpha} dy d\tau \\ &\leq \left\{ \|g\|_{\infty} \|f\|_{\alpha} (\|u_x\|_{\infty})^{\alpha} + \|f\|_{\infty} \|g\|_{\alpha} (\|v_x\|_{\infty})^{\alpha} \right\} \frac{C_3(\alpha)}{\alpha} t^{\alpha/2} \\ &\leq \left\{ \|g\|_{\infty} \|f\|_{\alpha} \left( \|\psi_x\| + C_2(0) \sqrt{t} \|f\|_{\infty} \|g\|_{\infty} \right)^{\alpha} \right\} \end{aligned}$$

$$+ \|f\|_{\infty} \|g\|_{\infty} \left\{ \|\phi_x\| + C_2(0) \sqrt{t} \|f\|_{\infty} \|g\|_{\infty} \right\}^{\alpha} \frac{C_3(\alpha)}{\alpha} t^{\alpha/2} \quad (2.6.43)$$

So  $v_{xx} - \phi_{xx} \rightarrow 0$  like  $O(t^{\alpha/2})$  if  $\|f\|_{\infty} \leq E$

$$\Rightarrow \|v_{xx}\|_{\infty} \cong \|\phi_{xx}\|_{\infty} \cong \|v''_0\|_{\infty} \quad (2.6.44)$$

Now we turn to estimate  $v_t$ .

Since

$$\begin{aligned} v &= \phi + \int_0^t \int_0^{\infty} K(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau \\ &\Rightarrow v_t = \phi_t + \frac{\partial}{\partial t} \int_0^t F(t, \tau) d\tau \\ &= \phi_t + F(t, t) + \int_0^t \frac{\partial F(t, \tau)}{\partial t} d\tau \end{aligned} \quad (2.6.45)$$

Where

$$F(t, \tau) = \int_0^{\infty} K(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau$$

Using the Causal Property of the Fundamental Solution  $K$  (STAKGOLD [111]) we say  $F(t, t) = 0$ . Since  $K(x, y, t) = 0$  for  $t \leq 0$ , or in other words,  $K(x, y, t-\tau) = 0$  for  $t \leq \tau$ , differentiating  $F(t, \tau)$ .

$$F(t, \tau) = \int_0^{\infty} K(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau$$

We get

$$\frac{\partial F(t, \tau)}{\partial t} = \int_0^{\infty} K_t(x, y, t-\tau) \left\{ -f(u(y, \tau)) g(v(y, \tau)) \right\} dy d\tau$$

Using the identity property (2.6.7), the above expression can again be written as

$$\begin{aligned} \frac{\partial F(t, \tau)}{\partial t} = & \int_0^\infty K_t(x, y, t-\tau) \left[ g(v(y, \tau)) \left\{ f(u(x, \tau)) - f(u(y, \tau)) \right\} \right. \\ & \left. + f(u(x, \tau)) \left\{ g(v(x, \tau)) - g(v(y, \tau)) \right\} \right] dy d\tau \end{aligned}$$

Taking modulus on both the sides, one gets

$$\begin{aligned} & \left| \frac{\partial F(t, \tau)}{\partial t} \right| \\ & \leq \left\{ |g|_\infty |f|_\alpha (|u_x|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_x|_\infty)^\alpha \right\} \int_0^\infty |K_t(x, y, t-\tau)| |y-x|^\alpha dy \end{aligned}$$

Now, integrating both the sides w.r.t. to  $\tau$  and considering the limits from 0 to  $t$ , the right hand side will be given the value of the domain constant as calculated in Appendix A, (A9).

$$\begin{aligned} & \int_0^t \left| \frac{\partial F}{\partial t}(t, \tau) \right| d\tau \\ & \leq \left\{ |g|_\infty |f|_\alpha (|u_x|_\infty)^\alpha + |f|_\infty |g|_\alpha (|v_x|_\infty)^\alpha \right\} \times \frac{C_5(\alpha)}{\alpha} t^{\alpha/2} \end{aligned}$$

As  $t \rightarrow 0$ , since all the other terms are bounded above, the L. H. S. goes to zero.

$$\Rightarrow |v_t - \phi_t| \rightarrow 0 \text{ as } t \rightarrow 0.$$

Consequently,

$$\Rightarrow \left[ \frac{|v_t|}{\inf |\theta'(t)|} \right]^\alpha \leq \left[ \frac{|\phi_{xx}|}{\inf |\theta'(t)|} \right]^\alpha \cong \left[ \frac{|v_0''|}{\inf |\theta'(0)|} \right]^\alpha \quad (2.6.46)$$

Now, in view of the inequalities (2.6.42), (2.6.44) and (2.6.46) we can say that these estimates can be made very small by choosing flat initial data  $|u_0''| \ll 1$  and  $|v_0''| \ll 1$ .

Thus by taking  $t$  small enough and initial data flat enough,  $M_1$  and  $M_2$  of (2.6.33) can be made sufficiently small.

$$\begin{aligned}
 \|\tilde{f}\|_{\alpha} &\leq M_1 \|\theta' - \psi_t\|_{\alpha} + M_2 \|f\|_{\alpha} \\
 &= M_1 \|\theta' - \psi_t\|_{\alpha} + \frac{C(\alpha)}{\alpha(1-\alpha)} \frac{(\inf |g|^2)^{-1}}{(\inf |\theta'|)^{\alpha}} |g|_{\infty} \\
 &\quad \left\{ |g|_{\infty} (|u_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} \|f\|_{\alpha} \\
 &\leq M_1 \|\theta' - \psi_t\|_{\alpha} + \frac{C(\alpha)}{\alpha(1-\alpha)} \left\{ |g|_{\infty} (|u_0''|_{\infty})^{\alpha} + |g|_{\alpha} (|v_0''|_{\infty})^{\alpha} \right\} E \\
 &\leq E
 \end{aligned} \tag{2.6.47}$$

In view of the smallness of  $M_2$ , a choice of  $E$  can be found so that  $\|\tilde{f}\|_{\alpha} \leq E$  if  $\|f\|_{\alpha} \leq E$ .

Hence  $T_{\theta} : B_E \subset C^{\alpha} \longrightarrow B_E \subset C^{\alpha}$ . ■

Finally, we note that  $T_{\theta}$  is a continuous map from  $C^{\alpha}$  to  $C^{\alpha}$ . This follows from the continuous dependence of the solution "u" of the direct problem on the right hand side function  $f$ .

**LEMMA 2 :** The set  $U = \left\{ f : \|f\|_{\infty} \leq E, |f|_{\alpha} \leq E \right\}$  is closed set in  $C^{\beta}$  for  $0 < \beta < \alpha < 1$ .

**PROOF :** Clearly,  $U$  is a closed set in  $C^{\alpha}$ .

Let  $\left\{ f_n \right\}$  be a sequence in  $U$  such that  $f_n \longrightarrow f$  in  $C^{\beta}$  for  $\beta < \alpha$ .

We claim that  $f \in U$ .

Since  $f_n \longrightarrow f$  in  $C^{\beta}$

$$\Rightarrow \|f_n - f\|_\infty \rightarrow 0$$

$$\text{So } \|f\|_\infty \leq \|f_n\|_\infty + \|f_n - f\|_\infty$$

$$\Rightarrow \|f\|_\infty \leq \overline{\lim} \left( \|f_n\|_\infty + \|f_n - f\|_\infty \right) \leq E.$$

$$\begin{aligned} \text{Further, } \frac{|f(x) - f(y)|}{|x - y|^\alpha} &= \frac{|\lim (f_n(x) - f_n(y))|}{|x - y|^\alpha} \\ &\leq \overline{\lim} \frac{|f_n(x) - f_n(y)|}{|x - y|^\alpha} \\ &\leq \overline{\lim} E = E \end{aligned}$$

$$\Rightarrow \|f\|_\alpha \leq E$$

Hence  $U$  is closed in  $C^\beta$ . ■

**THEOREM 2 (EXISTENCE)** : Assume that  $T_\theta$  is a self mapping from  $U$  into itself, where  $U = \left\{ f : \|f\|_\infty \leq E, \|f\|_\alpha \leq E \right\}$ . Then  $T_\theta$  has a fixed point.

**PROOF** : The set  $U$  is a closed, convex set in  $C^\alpha$ . By the above lemma  $U$  is also closed and convex set in  $C^\beta$ . Further, since  $U$  is a bounded set in  $C^\alpha$ ,  $T_\theta U$  is precompact in  $C^\beta$ . Hence  $T_\theta$  is a completely continuous operator taking  $T_\theta : U \subset C^\beta \rightarrow U \subset C^\beta$ . Therefore, by SCHAUDER's fixed point theorem (FRIEDMANN, p. 189 [49]),  $T_\theta$  has a fixed point in  $C^\beta$ . ■

**REMARK :**

However, the above theorem does not prove the existence of a limit of the iterated sequence  $\{f^{(k)}\}$  obtained above. In the next chapter we shall prove a centred contraction mapping Property of the  $T_\theta$  map under additional hypotheses

which will show the convergence of the iterates  $\{f^{(k)}\}$  and there by establish the convergence of the iterative scheme outlined above.

## 2.7 CONCLUSIONS :

The basic problem dealt with in this chapter is to prove the existence of a non-linear source function of a system of two Reaction-Diffusion equations in a semi-infinite spatial domain. At the out set the original problem is reformulated as a fixed point problem. An iterative scheme is outlined via a self mapping operator in a bounded set in the Hölder space. Using the properties of Green's function many domain estimates have been calculated explicitly. Finally making use of the SCHAUDER's fixed point theorem, the existence of a fixed point of the solution of the original source problem is established.



## CHAPTER - III

### UNIQUENESS OF THE SOLUTION OF THE REACTION-DIFFUSION SOURCE

#### PROBLEM AND CONVERGENCE OF AN ITERATIVE SCHEME

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#### 3.1 INTRODUCTION :

The existence of the solution of the Reaction-Diffusion source problem with superposed Dirichlet data is established in the previous chapter. In this chapter, the uniqueness of the solution and the convergence of the proposed iterative scheme outlined in the previous chapter is established. We follow the notations of chapter II and assume the validity of the conditions stated in that chapter.

The existence theorem ensures existence of a  $\theta$ -fixed point of the operator  $T_\theta$ . Therefore, there is an "f" such that

$$f(\theta(t)) = T_\theta[f](t) \quad (3.1.1)$$

In order to prove the uniqueness, we proceed as follows :

Denote 
$$\bar{p}(\theta(t)) = T_\theta[p](t) \quad (3.1.2)$$

We show that

$$\|f - \bar{p}\|_\alpha \leq \Lambda \|f - p\|_\alpha \quad \text{with } \Lambda \in (0,1)$$

Consequently, there can not be any other  $\theta$ -fixed point "p" of  $T_\theta$ , for in that

case

$$\|f - \bar{p}\|_{\alpha} \leq \Lambda \|f - p\|_{\alpha} \quad \text{which is a clear contradiction.}$$

Below we show that  $T_{\theta}$ , indeed satisfies the following contraction property i. e., there exists  $\Lambda$  satisfying  $0 \leq \Lambda < 1$  such that

$$\begin{aligned} \|f(\theta) - \bar{p}(\theta)\|_{\alpha} &= \|T_{\theta}[f] - T_{\theta}[p]\|_{\alpha} \\ &\leq \Lambda \|f - p\|_{\alpha} \end{aligned} \quad (3.1.3)$$

The main result of this chapter is stated in section 2. The sup and semi-norm are computed in sections 3 and 4 respectively to establish the central result. Finally a brief summary with possible future studies are posed in section 5.

### 3.2 THE THEOREM :

It is assumed that the overposed boundary function  $\theta$  belongs to the class of Lipschitz functions with some additional smoothness assumptions to be stated precisely later.

For any uniformly Lipschitz function  $f$  and any  $g \in C^{\alpha}$ ,  $f \circ g \in C^{\alpha}$ . In fact

$$\|f \circ g\|_{\alpha} \leq \|f\|_{\infty} + |f|_1 |g|_{\alpha} \quad (3.2.1)$$

We saw that for any Lipschitz class of functions on a bounded domain, preserves  $C^{\alpha}$  regularity under composition. This is not sufficient to show uniqueness. We say that a function  $f \in \text{Lip}$  has "property S" if

**PROPERTY S :** Given function  $u, v \in C^{\alpha}$ , for  $0 < \alpha < 1$ , the mapping  $u \longrightarrow f(u)$  is  $C^{\alpha}$ , that is there exist  $C_s < \infty$  and

$$\|f(u) - f(v)\|_{\alpha} \leq C_s \|u - v\|_{\alpha} \quad (3.2.2)$$

**THEOREM :** If  $T_{\theta}$  operator for the the overposed boundary value problem possesses a  $\theta$ -fixed point  $f$  and both  $f, g$  of R-D equations satisfy property S,

then  $f$  is the unique fixed point. Furthermore, the operator  $T_\theta$  is contractive in  $C^\alpha$  for  $\alpha < 1$  about  $f$ . i. e.,

$$\|T_\theta[f] - T_\theta[p]\|_\alpha \leq \Lambda \|f - p\|_\alpha \quad \text{with } 0 \leq \Lambda < 1.$$

### 3.3 SUP-NORM ESTIMATION :

Let  $u = u(x, t; f, g)$ ; be the solution of the differential equation

$$u_t - u_{xx} = f(u) g(v), \quad x > 0, t > 0; \quad (3.3.1a)$$

$$u_x(0, t) = 0, \quad t > 0; \quad (3.3.1b)$$

$$u(x, 0) = u_0(x), \quad x > 0; \quad (3.3.1c)$$

satisfying the overposed boundary condition,

$$u(0, t) = \theta(t), \quad t > 0; \quad (3.3.2)$$

where  $v$  is the solution of the differential equation

$$v_t - v_{xx} = -f(u) g(v), \quad x > 0, t > 0; \quad (3.3.3a)$$

$$v_x(0, t) = 0, \quad t > 0; \quad (3.3.3b)$$

$$v(x, 0) = v_0(x), \quad x > 0; \quad (3.3.3c)$$

Let  $p$  correspond to another set of solutions of the over-posed problem with  $w = w(x, t; p, g)$  and  $\tilde{v}(x, t; p, g)$  satisfying the following :

$$w_t - w_{xx} = p(w) g(\tilde{v}), \quad x > 0, t > 0; \quad (3.3.4a)$$

$$w_x(0, t) = 0, \quad t > 0; \quad (3.3.4b)$$

$$w(x, 0) = u_0(x), \quad x > 0; \quad (3.3.4c)$$

$$w(0, t) = \theta(t), \quad t > 0; \quad (3.3.4d)$$

and

$$\tilde{v}_t - \tilde{v}_{xx} = -p(w) g(\tilde{v}), \quad x > 0, t > 0; \quad (3.3.5a)$$

$$\tilde{v}_x(0, t) = 0, \quad t > 0; \quad (3.3.5b)$$

$$\tilde{v}(x, 0) = v_0(x), \quad x > 0; \quad (3.3.5c)$$

The expression for the map  $T_\theta$  from (2.6.13) is given explicitly by

$$f(\theta(t)) = T_\theta[f](t)$$

$$= \frac{\theta'(t) - \psi_t(0,t) - \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) f(u(y,\tau)) g(v(y,\tau)) dy d\tau}{g(v(0,t))} \quad (3.3.6)$$

and analogously,

$$\bar{p}(\theta(t)) = T_\theta[p](t)$$

$$= \frac{\theta'(t) - \psi_t(0,t) - \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) p(w(y,\tau)) g(\tilde{v}(y,\tau)) dy d\tau}{g(\tilde{v}(0,t))} \quad (3.3.7)$$

Here  $\psi$  is the comparison function (repeated here for convenience).

$$\begin{aligned} \psi_t - \psi_{xx} &= 0, & x > 0, t > 0; \\ \psi_x(0,t) &= 0, & t > 0; \\ \psi(x,0) &= u_0(x), & x > 0; \end{aligned} \quad (3.3.8)$$

First of all we set out to estimate  $\|f - \bar{p}\|_\infty$ , for which we use the expressions in (3.3.6) and (3.3.7). In order to avoid repetitive expressions we divide the expression for  $I_0$  below into several parts and estimate each part separately.

$$\begin{aligned} I_0 &= f(\theta(t)) g(v(0,t)) - \bar{p}(\theta(t)) g(\tilde{v}(0,t)) \\ &= \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) \left[ p(w(y,\tau)) g(\tilde{v}(y,\tau)) - f(u(y,\tau)) g(v(y,\tau)) \right] dy d\tau \end{aligned}$$

Hence,

$$\left( f(\theta(t)) - \bar{p}(\theta(t)) \right) g(v(0,t)) = \bar{p}(\theta(t)) \left( g(\tilde{v}(0,t)) - g(v(0,t)) \right)$$

$$\begin{aligned}
& + \int_0^t \int_0^\infty K_{xx}(0, y, t-\tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
& + \int_0^t \int_0^\infty K_{xx}(0, y, t-\tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
& + \int_0^t \int_0^\infty K_{xx}(0, y, t-\tau) \left[ g(\tilde{v}(y, \tau)) - g(v(y, \tau)) \right] f(u(y, \tau)) dy d\tau \\
& = I_1 + I_2 + I_3 + I_4 \tag{3.3.9}
\end{aligned}$$

In order to find an estimate for L. H. S. of equation (3.3.9),  $I_1$ , for  $i = 1, 2, 3$  and  $4$  are computed separately. In the expressions  $I_1, I_3$  and  $I_4$ ,  $f - p$  does not enter explicitly. But the arguments  $w, u, v$  and  $\tilde{v}$  can be expressed in terms of  $f$  and  $p$  as they are solutions of the corresponding non-homogeneous differential equations. To calculate the max-norm of L. H. S. the idea is to express  $I_1, I_3$  and  $I_4$  in the form of  $|w_x - u_x|$  and  $|\tilde{v}_x - v_x|$ . Since the latter quantities involve first space derivative of the Neumann function, the domain estimates (Appendix A) will not blow up. Later  $|u_x - w_x|$  and  $|v_x - \tilde{v}_x|$  are expressed through lower order terms and their weighted values are calculated. These modifications are needed because in  $I_3$  and  $I_4$  the kernel is the second derivative of Neumann function.

Since  $u$  and  $w$  are the solutions of the non-homogeneous differential equations (3.3.1) with (3.3.2) and (3.3.4) respectively, so the difference is given by (see CANNON [26])

$$w(x, t) - u(x, t) =$$

$$\int_0^t \int_0^\infty K(x, y, t-\tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \tag{3.3.10}$$

Similarly, the difference between the solutions of (3.3.3) and (3.3.5) is

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$$\tilde{v}(x,t) - v(x,t) =$$

$$\int_0^t \int_0^\infty K(x,y,t-\tau) \left[ f(u(y,\tau)) g(v(y,\tau)) - p(w(y,\tau)) g(\tilde{v}(y,\tau)) \right] dy d\tau \quad (3.3.11)$$

From (3.3.10) and (3.3.11) it is evident that  $u(x,t) - w(x,t) = \tilde{v}(x,t) - v(x,t)$  and hence  $|w - u| = |\tilde{v} - v|$  and the same is true for their higher order derivatives.

Since the domain integrals (Appendix A) have a form, where kernel is multiplied by  $|x - y|^\alpha$ , we require to use the identity property (2.6.7) defined in chapter II, to bring the expressions in the proper form.

The computation of  $\|f - \bar{p}\|_\infty$  involves several integral estimates. Therefore, it is subdivided into several steps.

**STEP 1 : ESTIMATION OF  $I_0$  ( $0 < \alpha < 1$ ).**

Let us compute  $I_2$ , first. Assume  $f, g \in C^\alpha$ . Using the identity property, it can be written as

$$\begin{aligned} I_2 = & \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) \left[ p(w(y,\tau)) - f(w(y,\tau)) \right. \\ & \left. - p(w(0,\tau)) + f(w(0,\tau)) \right] g(\tilde{v}(y,\tau)) dy d\tau \\ & + \int_0^t \int_0^\infty K_{xx}(0,y,t-\tau) \left[ p(w(0,\tau)) - f(w(0,\tau)) \right] \times \\ & \left[ g(\tilde{v}(y,\tau)) - g(\tilde{v}(0,\tau)) \right] dy d\tau \end{aligned}$$

Hence,

$$\begin{aligned} |I_2| \leq & \int_0^t \int_0^\infty |K_{xx}(0,y,t-\tau)| \|p - f\|_\alpha |w(y,\tau) - w(0,\tau)|^\alpha \|g\|_\infty dy d\tau \\ & + \int_0^t \int_0^\infty |K_{xx}(0,y,t-\tau)| \|p - f\|_\alpha |\tilde{v}(y,\tau) - \tilde{v}(0,\tau)|^\alpha \|g\|_\alpha dy d\tau \end{aligned}$$

Expand  $w$  and  $\tilde{v}$  by Taylor's expansion, considering centre at 0 and radius as  $y$ . From the assumptions that  $g_1 = 0 = g_2$  and Appendix A (A6),  $|I_2|$  becomes,

$$|I_2| \leq |g|_{\infty} \|p - f\|_{\alpha} (|w_{xx}|_{\infty})^{\alpha} \int_0^t \int_0^{\infty} |K_{xx}(0, y, t-\tau)| |(y^2/2)|^{\alpha} dy d\tau \\ + |g|_{\alpha} \|p - f\|_{\infty} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \int_0^t \int_0^{\infty} |K_{xx}(0, y, t-\tau)| |(y^2/2)|^{\alpha} dy d\tau$$

Hence,

$$|I_2| \leq \left\{ |g|_{\infty} \|p - f\|_{\alpha} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} \|p - f\|_{\infty} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right\} C_3(2\alpha) 2^{-(\alpha+2)} \frac{t^{\alpha}}{\alpha}$$

Noting that  $\alpha$ -semi norm and sup norm are individually less than  $\alpha$ -norm, we can write estimate  $I_2$  as

$$|I_2| \leq \left\{ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \right\} 2^{-(\alpha+2)} \frac{C_3(2\alpha)}{\alpha} t^{\alpha} \|p - f\|_{\alpha} \\ \leq \Lambda_1 t^{\alpha} \|p - f\|_{\alpha} \quad (3.3.12)$$

Where  $\Lambda_1 = \left\{ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} |\tilde{v}_{xx}|_{\infty}^{\alpha} \right\} \frac{C(\alpha)}{\alpha(1-\alpha)}$ ,  $C(\cdot)$  is defined in

chapter II (in the box). For the estimate of  $I_3$ , we use the "property S".

$$\text{Since, by definition } |u(\cdot, t)|_{\alpha} = \sup_{\xi \neq \eta} \frac{|u(\xi, t) - u(\eta, t)|}{|\xi - \eta|^{\alpha}}$$

Put  $\xi = y$ ,  $\eta = 0$  and  $u(\cdot, t) = f(w(\cdot, t)) - f(u(\cdot, t))$

then  $|f(w(\cdot, t)) - f(u(\cdot, t))|_1 =$

$$\sup_{y \neq 0} |f(w(y, t)) - f(u(y, t)) - f(w(0, t)) + f(u(0, t))| / |y|$$

Since  $f \in \text{Lip class}$  and satisfies property S, so making use of the identity property,  $I_3$  can be expressed in the following form.

$$\begin{aligned}
& + \int_0^t \int_0^\infty K_{xx}(0, y, t-\tau) \left[ g(\tilde{v}(0, \tau)) - g(v(0, \tau)) \right] \times \\
& \quad \left[ f(u(y, \tau)) - f(u(0, \tau)) \right] dy d\tau \\
& = I_{4,1} + I_{4,2}
\end{aligned} \tag{3.3.15}$$

Taking into account  $|w(x, t) - u(x, t)| = |v(\tilde{x}, t) - v(x, t)|$ , the fact that  $w(0, t) - u(0, t) = 0$  implies that  $I_{4,2} = 0$ . Thus,

$$\begin{aligned}
|I_4| & \leq |I_{4,1}| \leq \frac{1}{4} D_s |f|_\infty C_3(2) t |\tilde{v}_{xx} - v_{xx}|_\infty \\
& \leq \Lambda_3 t |u_{xx} - w_{xx}|_\infty
\end{aligned} \tag{3.3.16}$$

For  $g \in C^\alpha$  or Lip class the above argument is also true for  $I_1$ . Hence

$$I_1 \equiv 0 \tag{3.3.17}$$

Thus "step 1" leads to the following estimate in view of (3.3.12), (3.3.14), (3.3.16) and (3.3.17).

$$|f(\theta(t)) - \bar{p}(\theta(t))| |g(v(0, t))| \leq |I_1| + |I_2| + |I_3| + |I_4|$$

$$\begin{aligned}
\text{or} \quad & |f(\theta(t)) - \bar{p}(\theta(t))| \leq \left( \inf |g| \right)^{-1} \times \\
& \left[ \Lambda_1 t^\alpha \|p - f\|_\alpha + \Lambda_2 t |u_{xx} - w_{xx}|_\infty + \Lambda_3 t |u_{xx} - w_{xx}|_\infty \right] \\
\Rightarrow \quad & |f(\theta(t)) - \bar{p}(\theta(t))| \leq \left( \inf |g| \right)^{-1} \times \\
& \left[ \Lambda_1 t^\alpha \|p - f\|_\alpha + \Lambda_4 t |u_{xx} - w_{xx}|_\infty \right]
\end{aligned} \tag{3.3.18}$$

Where  $\Lambda_4 = \Lambda_2 + \Lambda_3$

**STEP 2 : ESTIMATION OF  $|w_{xx} - u_{xx}|$  :**

Differentiating (3.3.10) twice with respect to  $x$ , one gets

$$\begin{aligned}
& w_{xx}(x,t) - u_{xx}(x,t) \\
&= \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) \left[ p(w(y,\tau)) g(\tilde{v}(y,\tau)) - f(u(y,\tau)) g(v(y,\tau)) \right] dy d\tau \\
&= \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) \left[ p(w(y,\tau)) - f(w(y,\tau)) \right] g(\tilde{v}(y,\tau)) dy d\tau \\
&+ \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) \left[ f(w(y,\tau)) - f(u(y,\tau)) \right] g(\tilde{v}(y,\tau)) dy d\tau \\
&+ \int_0^t \int_0^\infty K_{xx}(x,y,t-\tau) \left[ g(\tilde{v}(y,\tau)) - g(v(y,\tau)) \right] f(u(y,\tau)) dy d\tau \\
&= I_5 + I_6 + I_7 \tag{3.3.19}
\end{aligned}$$

We note that, these expressions are similar to  $I_2$ ,  $I_3$  and  $I_4$  respectively (except that  $x$  stands in place of zero in  $K$ ). Now we provide estimates for these expressions in terms of the first derivative of  $u$  and  $w$ . Applying the identity property (2.6.7) with the assumptions  $f, g \in C^\alpha$  and Appendix A (A5).

$$\begin{aligned}
|I_5| &\leq \int_0^t \int_0^\infty |K_{xx}(x,y,t-\tau)| |p(w(y,\tau)) - f(w(y,\tau)) \\
&\quad - p(w(x,\tau)) + f(w(x,\tau))| |g(\tilde{v}(y,\tau))| dy d\tau \\
&+ \int_0^t \int_0^\infty |K_{xx}(x,y,t-\tau)| |p(w(x,\tau)) - f(w(x,\tau))| \times \\
&\quad |g(\tilde{v}(y,\tau)) - g(\tilde{v}(x,\tau))| dy d\tau \\
&\leq |g|_\infty |p - f|_\alpha (|w_x|_\infty)^\alpha \frac{C_3(\alpha)}{\alpha} t^{\alpha/2} \\
&+ |g|_\alpha |p - f|_\infty (|\tilde{v}_x|_\infty)^\alpha \frac{C_3(\alpha)}{\alpha} t^{\alpha/2} \\
&\leq \Lambda_5 \|p - f\|_\alpha t^{\alpha/2} + \Lambda_6 \|p - f\|_\alpha t^{\alpha/2} \\
&\leq \Lambda_7 \|p - f\|_\alpha t^{\alpha/2} \tag{3.3.20}
\end{aligned}$$

Since  $f \in \text{Lip}$  class and satisfies the property S,  $I_6$  is estimated as follows :

$$\begin{aligned}
I_6 &= \int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right. \\
&\quad \left. - f(w(x, \tau)) + f(u(x, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
&+ \int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) \left[ f(w(x, \tau)) - f(u(x, \tau)) \right] \times \\
&\quad \left[ g(\tilde{v}(y, \tau)) - g(\tilde{v}(x, \tau)) \right] dy d\tau \\
&= I_{6,1} + I_{6,2}
\end{aligned} \tag{3.3.21}$$

As  $f$  satisfies the property S,  $I_{6,1}$  is estimated as (3.3.13). Indeed,

$$\begin{aligned}
|I_{6,1}| &\leq \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |f(w) - f(u)|_1 |g|_\infty |x - y| dy d\tau \\
&\leq C_s \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |w(\cdot, \tau) - u(\cdot, \tau)|_1 |x - y| |g|_\infty dy d\tau \\
&\leq C_s |g|_\infty |u_x - w_x|_\infty \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |x - y| dy d\tau \\
&\leq \Lambda_8 |u_x - w_x|_\infty t^{1/2}
\end{aligned} \tag{3.3.22}$$

Further, since  $f, g \in \text{Lip class}$  with constants  $C$  and  $D$  (say) respectively, we have

$$\begin{aligned}
|I_{6,2}| &\leq \int_0^t \int_0^\infty \left\{ |K_{xx}(x, y, t-\tau)| |f|_{\text{lip}} |(w(x, \tau)) - (u(x, \tau))| |g|_{\text{lip}} \right. \\
&\quad \left. |x - y| (|\tilde{v}_x|_\infty) \right\} dy d\tau \\
&\leq C |w - u|_\infty D (|\tilde{v}_x|_\infty) \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |x - y| dy d\tau \\
&\leq \Lambda_9 |w - u|_\infty t^{1/2}
\end{aligned} \tag{3.3.23}$$

Hence from (3.3.22) and (3.3.23)

$$|I_6| \leq (\Lambda_8 |u_x - w_x|_\infty + \Lambda_9 |w - u|_\infty) t^{1/2} \tag{3.3.24}$$

$I_7$  is estimated exactly like  $I_6$ . So we can write

$$|I_7| \leq \Lambda_{10} |u_x - w_x|_{\infty} t^{1/2} + \Lambda_{11} |w - u|_{\infty} t^{1/2} \quad (3.3.25)$$

In view of (3.3.20), (3.3.24) and (3.3.25) Step 2 leads to the estimate

$$\begin{aligned} |w_{xx} - u_{xx}|_{\infty} &\leq |I_5| + |I_6| + |I_7| \\ &\leq \Lambda_7 \|p - f\|_{\alpha} t^{\alpha/2} + \Lambda_{12} |u_x - w_x|_{\infty} t^{1/2} + \Lambda_{13} |w - u|_{\infty} t^{1/2} \end{aligned} \quad (3.3.26)$$

Where  $\Lambda_{12} = \Lambda_8 + \Lambda_{10}$  and  $\Lambda_{13} = \Lambda_9 + \Lambda_{11}$

STEP 3 : ESTIMATION OF  $|w_x - u_x|$ .

This step is devoted to calculate  $|u_x - w_x|_{\infty}$ . Since

$$\begin{aligned} w_x(x, t) - u_x(x, t) &= \int_0^t \int_0^{\infty} K_x(x, y, t-\tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\ &+ \int_0^t \int_0^{\infty} K_x(x, y, t-\tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\ &+ \int_0^t \int_0^{\infty} K_x(x, y, t-\tau) \left[ g(\tilde{v}(y, \tau)) - g(v(y, \tau)) \right] f(u(y, \tau)) dy d\tau \\ &= I_8 + I_9 + I_{10} \end{aligned} \quad (3.3.27)$$

From (A3) of Appendix A, we get

$$\begin{aligned} |I_8| &\leq \int_0^t \int_0^{\infty} |K_x(x, y, t-\tau)| |p(w(y, \tau)) - f(w(y, \tau))| |g(\tilde{v}(y, \tau))| dy d\tau \\ &\leq \|p - f\|_{\infty} |g|_{\infty} \int_0^t \int_0^{\infty} |K_x(x, y, t-\tau)| dy d\tau \\ &\leq \Lambda_{14} \|p - f\|_{\alpha} t^{1/2} \end{aligned} \quad (3.3.28)$$

$$\Lambda_{14} = \left( C_2(0) |g|_{\infty} \right)$$

For  $f \in \text{Lip class}$

$$\begin{aligned}
 |I_9| &\leq \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| |f(w(y, \tau)) - f(u(y, \tau))| |g(\tilde{v}(y, \tau))| dy d\tau \\
 &\leq C \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| |w(y, \tau) - u(y, \tau)| |g(\tilde{v}(y, \tau))| dy d\tau \\
 &\leq C \|g\|_\infty C_2(0) \|w - u\|_\infty t^{1/2} \\
 &\leq \Lambda_{15} \|w - u\|_\infty t^{1/2}
 \end{aligned} \tag{3.3.29}$$

Now for  $g \in \text{Lip class}$ ,  $I_{10}$  is estimated similar to  $I_9$  and the result is given by

$$\begin{aligned}
 |I_{10}| &\leq D \|f\|_\infty C_2(0) \|\tilde{v} - v\|_\infty t^{1/2} \\
 &\leq \Lambda_{16} \|w - u\|_\infty t^{1/2}
 \end{aligned} \tag{3.3.30}$$

Therefore, in view of (3.3.28), (3.3.29) and (3.3.30) "step 3" gives the following estimate for  $w_x - u_x$  in terms of  $w - u$ :

$$\begin{aligned}
 \|w_x - u_x\|_\infty &\leq |I_8| + |I_9| + |I_{10}| \\
 &\leq \Lambda_{14} \|p - f\|_\infty t^{1/2} + \Lambda_{17} \|w - u\|_\infty t^{1/2}
 \end{aligned} \tag{3.3.31}$$

Where  $\Lambda_{17} = \Lambda_{15} + \Lambda_{16}$

**STEP 4 : ESTIMATION OF  $\|w - u\|_\infty$ .**

Finally, we estimate  $\|u - w\|_\infty$ , where the kernel is the Neumann function. Since we have calculated in Appendix A (A1) that the domain integral corresponding to  $K(x, y, t-\tau)$ , is a finite quantity, the estimation procedure will be easier. Expressing  $w - u$  in its integral form from (3.3.10)

$$w(x, t) - u(x, t)$$

$$\begin{aligned}
&= \int_0^t \int_0^\infty K(x, y, t-\tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
&+ \int_0^t \int_0^\infty K(x, y, t-\tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
&+ \int_0^t \int_0^\infty K(x, y, t-\tau) \left[ g(\tilde{v}(y, \tau)) - g(v(y, \tau)) \right] f(u(y, \tau)) dy d\tau \\
&= I_{11} + I_{12} + I_{13}
\end{aligned} \tag{3.3.32}$$

Let  $f, g \in C^\alpha$ , then

$$\begin{aligned}
|I_{11}| &\leq \int_0^t \int_0^\infty |K(x, y, t-\tau)| |p(w(y, \tau)) - f(w(y, \tau))| g(\tilde{v}(y, \tau)) dy d\tau \\
&\leq |p - f|_\infty |g|_\infty \int_0^t \int_0^\infty |K(x, y, t-\tau)| dy d\tau \\
&\leq |g|_\infty C_1(0) \|p - f\|_\alpha t
\end{aligned}$$

$$\Rightarrow |I_{11}| \leq \Lambda_{18} \|p - f\|_\alpha t \tag{3.3.33}$$

For  $f \in \text{Lip}$  with constant  $C$ ,  $I_{12}$  is estimated as

$$\begin{aligned}
|I_{12}| &\leq \int_0^t \int_0^\infty |K(x, y, t-\tau)| |f(w(y, \tau)) - f(u(y, \tau))| |g(\tilde{v}(y, \tau))| dy d\tau \\
&\leq C |g|_\infty C_1(0) |w - u|_\infty t \\
&\leq \Lambda_{19} |w - u|_\infty t
\end{aligned} \tag{3.3.34}$$

and finally, for  $g \in \text{Lip}$  class with constant  $D$ , we have

$$\begin{aligned}
|I_{13}| &\leq \int_0^t \int_0^\infty |K(x, y, t-\tau)| |g(\tilde{v}(y, \tau)) - g(v(y, \tau))| |f(u(y, \tau))| dy d\tau \\
&\leq D |f|_\infty C_1(0) |\tilde{v} - v|_\infty t \\
&\leq \Lambda_{20} |w - u|_\infty t
\end{aligned} \tag{3.3.35}$$

So the following estimate follows from "step 4".

$$\begin{aligned}
 |u - w|_{\infty} &\leq |I_{11}| + |I_{12}| + |I_{13}| \\
 &\leq \Lambda_{18} \|p - f\|_{\alpha} t + \Lambda_{21} |w - u|_{\infty} t
 \end{aligned}$$

Where  $\Lambda_{21} = \Lambda_{19} + \Lambda_{20}$

$$\Rightarrow |w - u|_{\infty} \leq \left( \frac{\Lambda_{18}}{1 - \Lambda_{21} t} \right) \|p - f\|_{\alpha} t \quad (3.3.36)$$

$$\text{Where } \Lambda_{22} = \left( \frac{\Lambda_{18}}{1 - \Lambda_{21} t} \right)$$

STEP 5 : ESTIMATION OF THE SUP-NORM OF  $f - \bar{p}$ .

Now substitute the inequality (3.3.36) in (3.3.31) to get

$$\begin{aligned}
 |w_x - u_x|_{\infty} &\leq \Lambda_{14} \|p - f\|_{\alpha} t^{1/2} + \Lambda_{17} \Lambda_{22} \|p - f\|_{\alpha} t^{3/2} \\
 &\leq \left( \Lambda_{14} + \Lambda_{17} \Lambda_{22} t \right) \|p - f\|_{\alpha} t^{1/2} \\
 &\leq \Lambda_{23} \|p - f\|_{\alpha} t^{1/2}
 \end{aligned} \quad (3.3.37)$$

Using the estimates (3.3.36) and (3.3.37), (3.3.26) can be written as

$$\begin{aligned}
 |w_{xx} - u_{xx}|_{\infty} &\leq \Lambda_7 \|p - f\|_{\alpha} t^{\alpha/2} + \Lambda_{12} |w_x - u_x| t^{1/2} + \Lambda_{13} |w - u|_{\infty} t^{1/2} \\
 &\leq \Lambda_7 \|p - f\|_{\alpha} t^{\alpha/2} + \Lambda_{12} \Lambda_{23} \|p - f\|_{\alpha} t + \Lambda_{13} \Lambda_{22} \|p - f\|_{\alpha} t^{3/2} \\
 &\leq \left( \Lambda_7 + \Lambda_{12} \Lambda_{23} t^{1-(\alpha/2)} + \Lambda_{13} \Lambda_{22} t^{(3-\alpha)/2} \right) \|p - f\|_{\alpha} t^{\alpha/2} \\
 &\leq \Lambda_{24} \|p - f\|_{\alpha} t^{\alpha/2}
 \end{aligned} \quad (3.3.38)$$

Finally substituting (3.3.38) in (3.3.18), we get

$$|f(\theta(t)) - \bar{p}(\theta(t))|$$

$$\leq \left( \inf |g| \right)^{-1} \left[ \Lambda_1 \|p - f\|_\alpha t^\alpha + \Lambda_4 \|u_{xx} - w_{xx}\|_\infty t \right]$$

$$\leq \left( \inf |g| \right)^{-1} \left[ \Lambda_1 \|p - f\|_\alpha t^\alpha + \Lambda_4 \Lambda_{24} \|p - f\| t^{(\alpha/2)+1} \right]$$

$$\leq \left( \inf |g| \right)^{-1} \left[ \Lambda_1 + \Lambda_4 \Lambda_{24} t^{1-(\alpha/2)} \right] \|p - f\|_\alpha t^\alpha$$

$$\leq \Lambda_{25} \|p - f\|_\alpha t^\alpha \quad (3.3.39)$$

$$\text{Where } \Lambda_{25} = \left( \inf |g| \right)^{-1} \left[ \Lambda_1 + \Lambda_4 \Lambda_{24} t^{1-(\alpha/2)} \right]$$

### 3.4 ESTIMATION OF HÖLDER SEMINORM :

For completion of  $\alpha$ -norm,  $\alpha$ -semi norm of  $(f - \bar{p})$  is to be estimated.

Consider  $t_1 > t_2$ . Then  $|f - \bar{p}|_\alpha$  is written as

$$|f - \bar{p}|_\alpha = \sup_{\theta(t_1) \neq \theta(t_2)} \frac{\left| \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right|}{|\theta(t_1) - \theta(t_2)|^\alpha} \quad (3.4.1)$$

Since  $f$  is a  $\theta$ -fixed point. So in view of (2.6.13) it is expanded corresponding to  $t_1$  and  $t_2$  explicitly as

$$\begin{aligned} f(\theta(t_1)) g(v(0, t_1)) &= \theta'(t_1) - \psi_t(0, t_1) \\ &\quad - \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \end{aligned} \quad (3.4.2)$$

$$f(\theta(t_2)) g(v(0, t_2)) = \theta'(t_2) - \psi_t(0, t_2)$$

$$- \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) f(u(y, \tau)) g(v(y, \tau)) dy d\tau \quad (3.4.3)$$

Similarly we can express for  $\bar{p}$

$$\begin{aligned} \bar{p}(\theta(t_1)) g(\tilde{v}(0, t_1)) &= \theta'(t_1) - \psi_t(0, t_1) \\ &- \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) p(w(y, \tau)) g(\tilde{v}(y, \tau)) dy d\tau \end{aligned} \quad (3.4.4)$$

$$\begin{aligned} \bar{p}(\theta(t_2)) g(\tilde{v}(0, t_2)) &= \theta'(t_2) - \psi_t(0, t_2) \\ &- \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) p(w(y, \tau)) g(\tilde{v}(y, \tau)) dy d\tau \end{aligned} \quad (3.4.5)$$

Subtracting equations (3.4.3) and (3.4.4) from (3.4.2) and then adding equation (3.4.5). The L. H. S expression of the equation becomes

$$\begin{aligned} &\left[ f(\theta(t_1)) g(v(0, t_1)) - \bar{p}(\theta(t_1)) g(\tilde{v}(0, t_1)) \right] \\ &- \left[ f(\theta(t_2)) g(v(0, t_2)) - \bar{p}(\theta(t_2)) g(\tilde{v}(0, t_2)) \right] \\ &= \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] g(v(0, t_1)) + \bar{p}(\theta(t_1)) \left[ g(v(0, t_1)) - g(\tilde{v}(0, t_1)) \right] \\ &- \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] g(v(0, t_2)) - \bar{p}(\theta(t_2)) \left[ g(v(0, t_2)) - g(\tilde{v}(0, t_2)) \right] \\ &= \left( \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right) g(v(0, t_1)) \\ &+ \bar{p}(\theta(t_1)) \left[ g(v(0, t_1)) - g(\tilde{v}(0, t_1)) \right] \\ &- \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \\ &- \bar{p}(\theta(t_2)) \left[ g(v(0, t_2)) - g(\tilde{v}(0, t_2)) \right] \end{aligned} \quad (3.4.6)$$

Similarly the R. H. S. becomes

$$\begin{aligned}
& \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
& - \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau
\end{aligned} \tag{3.4.7}$$

Hence from (3.4.6) and (3.4.7) one gets

$$\begin{aligned}
& \left( \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right) g(v(0, t_1)) \\
& + \bar{p}(\theta(t_1)) \left[ g(v(0, t_1)) - g(\tilde{v}(0, t_1)) \right] \\
& = \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \\
& + \bar{p}(\theta(t_2)) \left[ g(v(0, t_2)) - g(\tilde{v}(0, t_2)) \right] \\
& + \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
& - \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau
\end{aligned} \tag{3.4.8}$$

Since,  $|u(x, t) - w(x, t)| = |v(x, t) - \tilde{v}(x, t)|$  and at  $x = 0$ ,  $u(0, t) = w(0, t) = \theta(t)$ . We have for all  $t$ ,  $|v(0, t) - \tilde{v}(0, t)| = 0$ . Therefore, for  $g \in C^\infty$ ,

$$\bar{p}(\theta(t_1)) \left[ g(v(0, t_1)) - g(\tilde{v}(0, t_1)) \right] = 0 \tag{3.4.9}$$

and also

$$\bar{p}(\theta(t_2)) \left[ g(v(0, t_2)) - g(\tilde{v}(0, t_2)) \right] = 0 \tag{3.4.10}$$

Therefore, (3.4.8) simplifies to

$$\left( \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right) g(v(0, t_1))$$

$$\begin{aligned}
&= \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \\
&+ \left( \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \right. \\
&- \left. \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \right) \\
&= I_{14} + I_{15} \tag{3.4.11}
\end{aligned}$$

STEP 6 : DERIVATION OF ESTIMATE FOR  $I_{14} + I_{15}$  .

Since  $t_1 > t_2$  ,  $I_{15}$  is expressed equivalently as

$$\begin{aligned}
I_{15} &= \int_0^{t_2} \int_0^\infty \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) \right. \\
&\quad \left. - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
&+ \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
&= I_{15,1} + I_{15,2} \tag{3.4.12}
\end{aligned}$$

$I_{15,1}$  and  $I_{15,2}$  are estimated following earlier arguments. The very structure of  $I_{15,2}$  gives rise to three expressions analogous to  $I_2$ ,  $I_3$  and  $I_4$  except that " $t_i$ " ( $i = 1, 2$ ) stands in place of "0" and "t" in the integral.

$$\begin{aligned}
I_{15,2} &= \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
&+ \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\
&+ \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ g(\tilde{v}(y, \tau)) - g(v(y, \tau)) \right] f(u(y, \tau)) dy d\tau \\
&= I_{15,2,1} + I_{15,2,2} + I_{15,2,3} \tag{3.4.13}
\end{aligned}$$

Let  $f, g \in C^\alpha$ , for  $0 < \alpha < 1$ . Then,

$$\begin{aligned} I_{15,2,1} = & \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right. \\ & \left. - p(w(0, \tau)) + f(w(0, \tau)) \right] g(\tilde{v}(y, \tau)) dy d\tau \\ & + \int_{t_2}^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(0, \tau)) - f(w(0, \tau)) \right] \\ & \left[ g(\tilde{v}(y, \tau)) - g(\tilde{v}(0, \tau)) \right] dy d\tau \end{aligned}$$

Therefore,

$$\begin{aligned} |I_{15,2,1}| \leq & \left\{ |g|_\infty (|w_{xx}|_\infty)^\alpha \|p - f\|_\alpha + |g|_\alpha (|\tilde{v}_{xx}|_\infty)^\alpha \|p - f\|_\infty \right\} \times \\ & 2^{-(\alpha+2)} \frac{C_3(2\alpha)}{\alpha} |t_1 - t_2|^\alpha \\ \leq & \left\{ |g|_\infty (|w_{xx}|_\infty)^\alpha + |g|_\alpha (|\tilde{v}_{xx}|_\infty)^\alpha \right\} 2^{-(\alpha+2)} \frac{C_3(2\alpha)}{\alpha} \|p - f\|_\alpha |t_1 - t_2|^\alpha \\ \leq & \Lambda_1 \|p - f\|_\alpha |t_1 - t_2|^\alpha \end{aligned} \quad (3.4.14)$$

Let  $f \in \text{Lip class}$  and satisfy the property S. Then  $I_{15,2,2}$  is estimated analogous to  $I_3$ . The estimate can be seen to be

$$|I_{15,2,2}| \leq \Lambda_2 |u_{xx} - w_{xx}|_\infty |t_1 - t_2| \quad (3.4.15)$$

Similarly for  $g \in \text{Lip class}$  and satisfying the property S,  $I_{15,2,3}$  has the estimate

$$|I_{15,2,3}| \leq \Lambda_3 |w_{xx} - u_{xx}|_\infty |t_1 - t_2| \quad (3.4.16)$$

Therefore from (3.4.14), (3.4.15) and (3.4.16) we get

$$|I_{15,2}| \leq \Lambda_1 \|p - f\|_\alpha |t_1 - t_2|^\alpha + \Lambda_4 |w_{xx} - u_{xx}|_\infty |t_1 - t_2| \quad (3.4.17)$$

Before estimating  $I_{15,1}$ , we note that  $I_{15,1}$  can be rewritten as

$$\begin{aligned}
 I_{15,1} &= \int_0^t \int_0^\infty \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) \right. \\
 &\quad \left. - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
 &= \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ p(w(y, \tau)) g(\tilde{v}(y, \tau)) - f(u(y, \tau)) g(v(y, \tau)) \right] dy d\tau \\
 &= \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right] g(\tilde{v}(y, \tau)) dl dy d\tau \\
 &\quad + \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ f(w(y, \tau)) - f(u(y, \tau)) \right] g(\tilde{v}(y, \tau)) dl dy d\tau \\
 &\quad + \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ g(\tilde{v}(y, \tau)) - g(v(y, \tau)) \right] f(u(y, \tau)) dl dy d\tau \\
 &= I_{15,1,1} + I_{15,1,2} + I_{15,1,3} \tag{3.4.18}
 \end{aligned}$$

Below we estimate these three terms separately. Making use of the identity property and noting that  $f$ ,  $p$  and  $g$  are in  $C^\alpha$  for  $0 < \alpha < 1$ , we estimate  $I_{15,1,1}$  as follows.

$$\begin{aligned}
 I_{15,1,1} &= \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ p(w(y, \tau)) - f(w(y, \tau)) \right. \\
 &\quad \left. - p(w(0, \tau)) + f(w(0, \tau)) \right] g(\tilde{v}(y, \tau)) dl dy d\tau \\
 &\quad + \int_0^t \int_0^\infty \int_{t_2}^{t_1} K_{xxt}(0, y, l - \tau) \left[ f(w(0, \tau)) - p(w(0, \tau)) \right] \times \\
 &\quad \left[ g(\tilde{v}(y, \tau)) - g(\tilde{v}(0, \tau)) \right] dl dy d\tau \\
 |I_{15,1,1}| &\leq \int_0^t \int_0^\infty \int_{t_2}^{t_1} |K_{xxt}(0, y, l - \tau)| |p - f|_\alpha (|w_{xx}|_\infty)^\alpha \\
 &\quad \left| \frac{y^2}{2} \right|^\alpha |g|_\infty dl dy d\tau \\
 &\quad + \int_0^t \int_0^\infty \int_{t_2}^{t_1} |K_{xxt}(0, y, l - \tau)| |p - f|_\infty |g(\tilde{v}(y, \tau)) - g(\tilde{v}(0, \tau))| dl dy d\tau
 \end{aligned}$$

Hence,

$$\begin{aligned}
 |I_{15,1,1}| &\leq |p - f|_{\alpha} |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} \\
 &\quad \int_{t_2}^{t_1} \int_0^2 \int_0^{\infty} |K_{xxt}(0, y, 1-\tau)| \left| \frac{y^{2\alpha}}{2^{\alpha}} \right| dy d\tau dl \\
 &+ |p - f|_{\infty} (|\tilde{v}_{xx}|_{\infty})^{\alpha} |g|_{\alpha} \int_{t_2}^{t_1} \int_0^2 \int_0^{\infty} |K_{xxt}(0, y, 1-\tau)| \left| \frac{y^{2\alpha}}{2^{\alpha}} \right| dy d\tau dl
 \end{aligned}$$

Which by Appendix A (A8), gives

$$\begin{aligned}
 |I_{15,1,1}| &\leq 2^{-\alpha} \frac{C_4(2\alpha)}{\alpha(1-\alpha)} |p - f|_{\alpha} (|w_{xx}|_{\infty})^{\alpha} |g|_{\infty} |t_1 - t_2|^{\alpha} \\
 &+ 2^{-\alpha} \frac{C_4(2\alpha)}{\alpha(1-\alpha)} |p - f|_{\infty} (|\tilde{v}_{xx}|_{\infty})^{\alpha} |g|_{\alpha} |t_1 - t_2|^{\alpha} \\
 &\leq \Lambda_{26} \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} \quad (3.4.19)
 \end{aligned}$$

Where

$$\Lambda_{26} = 2^{-\alpha} \frac{C_4(2\alpha)}{\alpha(1-\alpha)} \left[ (|w_{xx}|_{\infty})^{\alpha} |g|_{\infty} + (|\tilde{v}_{xx}|_{\infty})^{\alpha} |g|_{\alpha} \right]$$

For  $f \in \text{Lip}$  class satisfying the property S. Following the steps of  $I_3$ ,  $I_{15,1,2}$  can be estimated as below :

$$\begin{aligned}
 |I_{15,1,2}| &\leq \int_0^{t_2} \int_0^{\infty} \int_{t_2}^{t_1} |K_{xxt}(0, y, 1-\tau)| |f(w(y, \tau)) - f(u(y, \tau))| \\
 &\quad |g(\tilde{v}(y, \tau))| dl dy d\tau \\
 &\leq C_s |u_{xx} - w_{xx}|_{\infty} |g|_{\infty} \int_0^{t_2} \int_0^{\infty} \int_{t_2}^{t_1} |K_{xxt}(0, y, 1-\tau)| y^2 dl dy d\tau \\
 &\leq C_s |g|_{\infty} C_4(2) |u_{xx} - w_{xx}|_{\infty} |t_1 - t_2| |\log |t_1 - t_2|| \\
 &\leq \Lambda_{27} |w_{xx} - u_{xx}|_{\infty} |t_1 - t_2| \quad (3.4.20)
 \end{aligned}$$

Where

$$\Lambda_{27} = C_s |g|_{\infty} C_4(2) |\log |t_1 - t_2||$$

For  $g \in \text{Lip}$  class and satisfying the property S, on similar line  $I_{15,1,3}$  can be estimated as :

$$\begin{aligned} |I_{15,1,3}| &\leq D_s |v_{xx} - \tilde{v}_{xx}| |f|_{\infty} \int_0^{t_2} \int_0^{\infty} \int_{t_2}^{t_1} |K_{xxt}(0, y, l-\tau)| y^2 dl dy d\tau \\ &\leq D_s C_4(2) |u_{xx} - w_{xx}|_{\infty} |f|_{\infty} |t_1 - t_2| |\log |t_1 - t_2|| \\ &\leq \Lambda_{28} |u_{xx} - w_{xx}|_{\infty} |t_1 - t_2| \end{aligned} \quad (3.4.21)$$

Since from (3.3.38)

$$|u_{xx} - w_{xx}|_{\infty} \leq \Lambda_{24} \|p - f\|_{\infty} t_1^{\alpha/2} \quad (3.4.22)$$

Hence substituting (3.4.22) in (3.4.20), (3.4.21) and (3.4.17) one gets

$$\begin{aligned} |I_{15,1,2}| &\leq \Lambda_{27} |u_{xx} - w_{xx}|_{\infty} |t_1 - t_2| \\ &\leq \Lambda_{27} \Lambda_{24} \|p - f\|_{\infty} t_1^{\alpha/2} |t_1 - t_2| \\ &\leq \Lambda_{29} \|p - f\|_{\infty} |t_1 - t_2| \end{aligned} \quad (3.4.23)$$

Where  $\Lambda_{29} = \Lambda_{27} \Lambda_{24} t_1^{\alpha/2}$

and similarly

$$\begin{aligned} |I_{15,1,3}| &\leq \Lambda_{28} \Lambda_{24} \|p - f\|_{\infty} t_1^{\alpha/2} |t_1 - t_2| \\ &\leq \Lambda_{30} \|p - f\|_{\infty} |t_1 - t_2| \end{aligned} \quad (3.4.24)$$

$$\begin{aligned} |I_{15,2}| &\leq \left( \Lambda_1 + \Lambda_4 \Lambda_{24} t_1^{\alpha} |t_1 - t_2|^{1-\alpha} \right) \|p - f\|_{\infty} |t_1 - t_2|^{\alpha} \\ &\leq \Lambda_{31} \|p - f\|_{\infty} |t_1 - t_2|^{\alpha} \end{aligned} \quad (3.4.25)$$

Similarly collecting (3.4.19), (3.4.23) and (3.4.24) we conclude

$$\begin{aligned}
 |I_{15,1}| &\leq |I_{15,1,1}| + |I_{15,1,2}| + |I_{15,1,3}| \\
 &\leq \Lambda_{26} \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} + \Lambda_{29} \|p - f\|_{\alpha} |t_1 - t_2| + \Lambda_{30} \|p - f\|_{\alpha} |t_1 - t_2| \\
 &\leq \left( \Lambda_{26} + \Lambda_{29} |t_1 - t_2|^{1-\alpha} + \Lambda_{30} |t_1 - t_2|^{1-\alpha} \right) \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} \\
 &\leq \Lambda_{32} \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} \tag{3.4.26}
 \end{aligned}$$

In view of (3.4.25) and (3.4.26), the following estimate of  $I_{15}$  is clear :

$$\begin{aligned}
 |I_{15}| &= \left| \int_0^{t_1} \int_0^{\infty} K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau))g(v(y, \tau)) - f(u(y, \tau))g(v(y, \tau)) \right] dy d\tau \right. \\
 &\quad \left. - \int_0^{t_2} \int_0^{\infty} K_{xx}(0, y, t_2 - \tau) \left[ p(w(y, \tau))g(v(y, \tau)) - f(u(y, \tau))g(v(y, \tau)) \right] dy d\tau \right| \\
 &\leq |I_{15,1}| + |I_{15,2}| \\
 &\leq \left( \Lambda_{32} + \Lambda_{31} \right) \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} \\
 &\leq \Lambda_{33} \|p - f\|_{\alpha} |t_1 - t_2|^{\alpha} \tag{3.4.27}
 \end{aligned}$$

Now coming to the estimate of  $I_{14}$ , given by

$$I_{14} = \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right]$$

We have, for  $g \in C^{\alpha}$

$$\begin{aligned}
 |I_{14}| &\leq |g|_{\alpha} |v(0, t_2) - v(0, t_1)|^{\alpha} |f(\theta(t_2)) - \bar{p}(\theta(t_2))| \\
 &\leq |g|_{\alpha} (|v_t|_{\infty})^{\alpha} |f - \bar{p}|_{\infty} |t_1 - t_2|^{\alpha} \\
 &\leq |g|_{\alpha} (|v_t|_{\infty})^{\alpha} \|f - \bar{p}\|_{\alpha} |t_1 - t_2|^{\alpha} \\
 &\leq \Lambda_{34} \|f - \bar{p}\|_{\alpha} |t_1 - t_2|^{\alpha} \tag{3.4.28}
 \end{aligned}$$

STEP 7 : ESTIMATION OF  $\alpha$ -SEMINORM OF  $f - \bar{p}$  :

$$\begin{aligned}
 & \left| \left( \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right) \right| |g(v(0, t_1))| \\
 & \leq \left| \int_0^{t_1} \int_0^\infty K_{xx}(0, y, t_1 - \tau) \left[ p(w(y, \tau))g(\tilde{v}(y, \tau)) - f(u(y, \tau))g(v(y, \tau)) \right] dy d\tau \right. \\
 & \quad \left. - \int_0^{t_2} \int_0^\infty K_{xx}(0, y, t_2 - \tau) \left[ p(w(y, \tau))g(\tilde{v}(y, \tau)) - f(u(y, \tau))g(v(y, \tau)) \right] dy d\tau \right| \\
 & \quad + \left| \left[ g(v(0, t_2)) - g(v(0, t_1)) \right] \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right| \\
 & \Rightarrow \left| \left( \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right) \right| |g|_\infty \\
 & \leq \Lambda_{33} \|p - f\|_\alpha |t_1 - t_2|^\alpha + \Lambda_{34} \|f - \bar{p}\|_\alpha |t_1 - t_2|^\alpha \tag{3.4.29}
 \end{aligned}$$

Now dividing both the sides of inequality (3.4.29) by  $|\theta(t_1) - \theta(t_2)|^\alpha$ , the  $\alpha$ -semi norm of  $|f - \bar{p}|_\alpha$  will be obtained

$$\begin{aligned}
 |f - \bar{p}|_\alpha &= \sup_{\theta(t_1) \neq \theta(t_2)} \frac{\left| \left[ f(\theta(t_1)) - \bar{p}(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \bar{p}(\theta(t_2)) \right] \right|}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 &\leq \left( \inf |g| \right)^{-1} \sup_{\theta(t_1) \neq \theta(t_2)} \left\{ \Lambda_{33} \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha} \|p - f\|_\alpha \right. \\
 &\quad \left. + \Lambda_{34} \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha} \|f - \bar{p}\|_\alpha \right\}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 |f - \bar{p}|_\alpha &\leq \left( \inf |g| \right)^{-1} \left\{ \Lambda_{33} \|p - f\|_\alpha + \Lambda_{34} \|f - \bar{p}\|_\alpha \right\} \left( \inf |\theta'| \right)^{-\alpha} \\
 &\leq \Lambda_{35} \|p - f\|_\alpha + \Lambda_{36} \|f - \bar{p}\|_\alpha \tag{3.4.30}
 \end{aligned}$$

We recall the expression (3.3.39) and note that

$$|f - \bar{p}|_{\infty} \leq \Lambda_{25} \|p - f\|_{\alpha} t^{\alpha}$$

and

$$|f - \bar{p}|_{\alpha} \leq \Lambda_{35} \|p - f\|_{\alpha} + \Lambda_{36} |f - \bar{p}|_{\alpha}$$

Now adding up the two estimates above, we obtain  $\alpha$ -norm of  $|f - \bar{p}|_{\alpha}$ .

Hence,

$$\begin{aligned} \|f - \bar{p}\|_{\alpha} &= |f - \bar{p}|_{\infty} + |f - \bar{p}|_{\alpha} \\ &\leq \Lambda_{25} \|p - f\|_{\alpha} t^{\alpha} + \Lambda_{35} \|p - f\|_{\alpha} + \Lambda_{36} |f - \bar{p}|_{\alpha} \\ &\leq \left( \Lambda_{25} t^{\alpha} + \Lambda_{35} \right) \|p - f\|_{\alpha} + \Lambda_{36} |f - \bar{p}|_{\alpha} \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - \bar{p}\|_{\alpha} &\leq \left( \frac{\Lambda_{25} t^{\alpha} + \Lambda_{35}}{1 - \Lambda_{36}} \right) \|p - f\|_{\alpha} \\ &\leq \Lambda_{37} \|p - f\|_{\alpha} \end{aligned} \tag{3.4.31}$$

Thus we have got an inequality where  $\alpha$ -norm of  $(f - \bar{p})$  is bounded by the  $\alpha$ -norm of  $(f - p)$ . Therefore, we reach a contradiction to the assumptions of the existence of two distinct solutions provided  $0 \leq \Lambda_{37} < 1$ .

**STEP 8 : ESTIMATE OF  $\Lambda_{37}$  :**

$$\text{Define } \Lambda_{38} = \Lambda_{25} t^{\alpha} + \Lambda_{35} + \Lambda_{36}$$

If we said that  $\Lambda_{38} < 1$  then  $\Lambda_{25} t^{\alpha} + \Lambda_{35} < 1 - \Lambda_{36}$  provided  $\Lambda_{36} < 1$ .

$$\text{So } \Lambda_{37} = \frac{\Lambda_{25} t^{\alpha} + \Lambda_{35}}{1 - \Lambda_{36}} \quad \text{is equivalent to } \Lambda_{38} < 1 .$$

For convenience  $\Lambda_i$ ,  $i = 1, \dots, 37$  are rewritten to estimate  $\Lambda_{38}$ .

$$\Lambda_1 = \left[ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right] \frac{C(\alpha)}{\alpha(1-\alpha)}$$

$$\Lambda_2 = \frac{1}{4} C_s |g|_{\infty} C_3(2)$$

$$\Lambda_3 = \frac{1}{4} D_s |f|_{\infty} C_3(2)$$

$$\Lambda_4 = \Lambda_2 + \Lambda_3 = \frac{1}{4} C_3(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right]$$

$$\Lambda_5 = |g|_{\infty} (|w_x|_{\infty})^{\alpha} \frac{C_3(\alpha)}{\alpha}$$

$$\Lambda_6 = |g|_{\alpha} (|\tilde{v}_x|_{\infty})^{\alpha} \frac{C_3(\alpha)}{\alpha}$$

$$\Lambda_7 = \Lambda_5 + \Lambda_6 = \frac{C_3(\alpha)}{\alpha} \left[ |g|_{\infty} (|w_x|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_x|_{\infty})^{\alpha} \right]$$

$$\Lambda_8 = C_s |g|_{\infty} C_3(1)$$

$$\Lambda_9 = C D (|\tilde{v}_x|_{\infty}) C_3(1)$$

$$\Lambda_{10} = D_s |f|_{\infty} C_3(1)$$

$$\Lambda_{11} = C D (|u_x|_{\infty}) C_3(1)$$

$$\Lambda_{12} = \Lambda_8 + \Lambda_{10} = C_3(1) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right]$$

$$\Lambda_{13} = \Lambda_9 + \Lambda_{11} = C D C_3(1) \left[ (|\tilde{v}_x|_{\infty}) + (|u_x|_{\infty}) \right]$$

$$\Lambda_{14} = C_2(0) |g|_{\infty}$$

$$\Lambda_{15} = C |g|_{\infty} C_2(0)$$

$$\Lambda_{16} = D |f|_{\infty} C_2(0)$$

$$\Lambda_{17} = \Lambda_{15} + \Lambda_{16} = \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_2(0)$$

$$\Lambda_{18} = |g|_{\infty} C_1(0)$$

$$\Lambda_{19} = C |g|_{\infty} C_1(0)$$

$$\Lambda_{20} = D |f|_{\infty} C_1(0)$$

$$\Lambda_{21} = \Lambda_{19} + \Lambda_{20} = \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0)$$

$$\Lambda_{22} = \frac{\Lambda_{18}}{1 - \Lambda_{21} t} = \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t}$$

$$\begin{aligned} \Lambda_{23} &= \Lambda_{14} + \Lambda_{17} \Lambda_{22} t \\ &= C_2(0) |g|_{\infty} + t \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_2(0) \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t} \end{aligned}$$

$$\begin{aligned} \Lambda_{24} &= \Lambda_7 + \Lambda_{12} \Lambda_{23} t^{1-(\alpha/2)} + \Lambda_{13} \Lambda_{22} t^{(3-\alpha)/2} \\ &= \frac{C_3(\alpha)}{\alpha} \left[ |g|_{\infty} (|w_x|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_x|_{\infty})^{\alpha} \right] + \end{aligned}$$

$$C_3(1) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] t^{1-(\alpha/2)}$$

$$\left[ C_2(0) |g|_{\infty} + t \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_2(0) \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t} \right]$$

$$+ C D C_3(1) \left[ (|\tilde{v}_x|_{\infty}) + (|u_x|_{\infty}) \right] t^{(3-\alpha)/2} \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t}$$

$$\Lambda_{25} = \left( \inf |g| \right)^{-1} \left[ \Lambda_1 + \Lambda_4 \Lambda_{24} t^{1-(\alpha/2)} \right]$$

$$= \left( \inf |g| \right)^{-1} \left\{ \left[ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right] \frac{C(\alpha)}{\alpha(1-\alpha)} \right.$$

$$\left. + \frac{1}{4} C_3(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] \Lambda_{24} t^{1-(\alpha/2)} \right\}$$

$$\Lambda_{26} = \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right]$$

$$\Lambda_{27} = C_s |g|_{\infty} C_4(2) |\log |t_1 - t_2||$$

$$\Lambda_{28} = D_s |f|_{\infty} C_4(2) |\log |t_1 - t_2||$$

$$\Lambda_{29} = \Lambda_{27} \Lambda_{24} t_1^{\alpha/2} = C_s |g|_{\infty} C_4(2) |\log |t_1 - t_2|| t_1^{\alpha/2} \Lambda_{24}$$

$$\Lambda_{30} = \Lambda_{28} \Lambda_{24} t_1^{\alpha/2} = D_s |f|_{\infty} C_4(2) |\log |t_1 - t_2|| t_1^{\alpha/2} \Lambda_{24}$$

$$\Lambda_{31} = \Lambda_1 + \Lambda_4 \Lambda_{24} t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha}$$

$$= \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right]$$

$$+ \frac{1}{4} C_3(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha} \Lambda_{24}$$

$$\Lambda_{32} = \Lambda_{26} + \Lambda_{29} |t_1 - t_2|^{1-\alpha} + \Lambda_{30} |t_1 - t_2|^{1-\alpha}$$

$$= \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\infty} (|w_{xx}|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} \right]$$

$$+ \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] |\log |t_1 - t_2|| t_1^{\alpha/2} C_4(2) |t_1 - t_2|^{1-\alpha} \Lambda_{24}$$

$$\Lambda_{33} = \Lambda_{31} + \Lambda_{32}$$

$$\Lambda_{34} = |g|_{\alpha} (|v_t|_{\infty})^{\alpha}$$

$$\Lambda_{35} = \left( \inf |g| \right)^{-1} \frac{\Lambda_{33}}{(\inf |\theta'|)^{\alpha}}$$

$$\Lambda_{36} = \left( \inf |g| \right)^{-1} \frac{\Lambda_{34}}{(\inf |\theta'|)^{\alpha}} = (\inf |g|)^{-1} |g|_{\alpha} \frac{(|v_t|_{\infty})^{\alpha}}{(\inf |\theta'|)^{\alpha}}$$

$$\Lambda_{37} = \frac{\Lambda_{25} t^{\alpha} + \Lambda_{35}}{1 - \Lambda_{36}}$$

Now one can estimate the the following

$$\begin{aligned} \Lambda_{38} &= \Lambda_{36} + \Lambda_{35} + \Lambda_{25} t^{\alpha} \\ &= (\inf |g|)^{-1} |g|_{\alpha} \frac{(|v_t|_{\infty})^{\alpha}}{(\inf |\theta'|)^{\alpha}} \\ &\quad + \frac{(\inf |g|)^{-1}}{(\inf |\theta'|)^{\alpha}} \left\{ 2 \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} + (|w_{xx}|_{\infty})^{\alpha} |g|_{\infty} \right] \right. \\ &\quad + \frac{1}{4} C_3(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha} \Lambda_{24} \\ &\quad + C_4(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] |\log |t_1 - t_2|| t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha} \Lambda_{24} \left. \right\} \\ &\quad + (\inf |g|)^{-1} t^{\alpha} \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} + (|w_{xx}|_{\infty})^{\alpha} |g|_{\infty} \right] \\ &\quad + \frac{1}{4} C_3(2) (\inf |g|)^{-1} \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] \Lambda_{24} t^{1+(\alpha/2)} \\ &= (\inf |g|)^{-1} |g|_{\alpha} \frac{(|v_t|_{\infty})^{\alpha}}{(\inf |\theta'|)^{\alpha}} \\ &\quad + \frac{1}{4} C_3(2) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha} \frac{(\inf |g|)^{-1}}{(\inf |\theta'|)^{\alpha}} \Lambda_{24} \end{aligned}$$

$$\begin{aligned}
& + \left[ 2 \frac{(\inf |g|)^{-1}}{(\inf |\theta'|)^{\alpha}} + (\inf |g|)^{-1} t^{\alpha} \right] \\
& \times \frac{C(\alpha)}{\alpha(1-\alpha)} \left[ |g|_{\alpha} (|\tilde{v}_{xx}|_{\infty})^{\alpha} + (|w_{xx}|_{\infty})^{\alpha} |g|_{\infty} \right] \\
& + \left[ \frac{(\inf |g|)^{-1}}{(\inf |\theta'|)^{\alpha}} C_4(2) |\log |t_1 - t_2|| t_1^{\alpha/2} |t_1 - t_2|^{1-\alpha} \right. \\
& \left. + (\inf |g|)^{-1} t^{1+(\alpha/2)} \frac{1}{4} C_3(2) \right] \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] \Lambda_{24}
\end{aligned} \tag{3.4.32}$$

Let coefficient of  $\Lambda_{24}$  be  $\delta_1(t)$ . Now calculate  $\delta_1(t) \Lambda_{24}$ .

$$\begin{aligned}
\delta_1(t) \Lambda_{24} &= \left[ \delta_1(t) \frac{C_3(\alpha)}{\alpha} \left[ |g|_{\infty} (|w_x|_{\infty})^{\alpha} + |g|_{\alpha} (|\tilde{v}_x|_{\infty})^{\alpha} \right] \right. \\
& \quad \left. + \delta_1(t) \left[ C_s |g|_{\infty} + D_s |f|_{\infty} \right] C_3(1) t^{1-(\alpha/2)} \right. \\
& \quad \times \left[ C_2(0) |g|_{\infty} + t \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_2(0) \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t} \right] \\
& \quad + \delta_1(t) \left[ (|\tilde{v}_x|_{\infty}) + (|u_x|_{\infty}) \right] C D C_3(1) t^{(3-\alpha)/2} \frac{|g|_{\infty} C_1(0)}{1 - \left[ C |g|_{\infty} + D |f|_{\infty} \right] C_1(0) t}
\end{aligned} \tag{3.4.33}$$

Represent the second term of R. H. S. of (3.4.33) as  $\delta_2(t)$ . Now substitute the above expression in (3.4.32) and with some suitable arrangement this can be written as

$$\begin{aligned}
\Lambda_{38} &= (\inf |g|)^{-1} |g|_{\infty} \frac{(|v_t|_{\infty})^{\alpha}}{(\inf |\theta'|)^{\alpha}} + \delta_2(t) \\
&+ (\inf |g|)^{-1} \left[ 2 + (\inf |\theta'|)^{\alpha} t^{\alpha} \right] \frac{C(\alpha)}{\alpha(1-\alpha)} |g|_{\infty} \left( \frac{|w_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha} \\
&+ (\inf |g|)^{-1} \left[ 2 + (\inf |\theta'|)^{\alpha} t^{\alpha} \right] \frac{C(\alpha)}{\alpha(1-\alpha)} |g|_{\infty} \left( \frac{|\tilde{v}_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha} \\
&+ \delta_3(t) \left( \frac{|w_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha} + \delta_4(t) \left( \frac{|\tilde{v}_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha} + \delta_5(t) \left( \frac{|u_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha}
\end{aligned}$$

Where  $\delta_3(t)$ ,  $\delta_4(t)$  and  $\delta_5(t)$  are the coefficients obtained with some suitable arrangements.

So it is enough to show,  $\left( \frac{|v_t|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$ ,  $\left( \frac{|w_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$ ,  $\left( \frac{|\tilde{v}_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$ ,  $\left( \frac{|w_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$ ,  $\left( \frac{|\tilde{v}_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$  and  $\left( \frac{|u_x|_{\infty}}{\inf |\theta'|} \right)^{\alpha}$  are bounded. Utilizing the

representation of the previous chapter

$$\begin{aligned}
w_{xx}(x,t) &= \psi_{xx}(x,t) + \int_0^t \int_0^{\infty} K_{xx}(x,y,t-\tau) p(w(y,\tau)) g(\tilde{v}(y,\tau)) dy d\tau \\
&= \psi_{xx}(x,t) + \int_0^t \int_0^{\infty} K_{xx}(x,y,t-\tau) \left[ g(\tilde{v}(y,\tau)) \left\{ p(w(y,\tau)) - p(w(x,\tau)) \right\} \right. \\
&\quad \left. + p(w(x,\tau)) \left\{ g(\tilde{v}(y,\tau)) - g(\tilde{v}(x,\tau)) \right\} \right] dy d\tau
\end{aligned}$$

$$\Rightarrow |w_{xx} - \psi_{xx}| \leq \left[ |g|_{\infty} |p|_{\infty} (|w_x|_{\infty})^{\alpha} + |p|_{\infty} |g|_{\infty} (|\tilde{v}_x|_{\infty})^{\alpha} \right] \frac{C_3(\alpha)}{\alpha} t^{\alpha/2}$$

and

$$\begin{aligned}
 |w_x - \psi_x| &\leq \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| |p(w(y, \tau))| |g(\tilde{v}(y, \tau))| dy d\tau \\
 &\leq C_2(0) \sqrt{t} |p|_\infty |g|_\infty
 \end{aligned} \tag{3.4.35}$$

Similarly,  $\tilde{v}$  can be represented as

$$\begin{aligned}
 \tilde{v}(x, t) &= \phi(x, t) + \int_0^t \int_0^\infty K(x, y, t-\tau) \left[ -p(w(y, \tau)) g(\tilde{v}(y, \tau)) \right] dy d\tau \\
 \Rightarrow |\tilde{v}_x - \phi_x| &\leq |p|_\infty |g|_\infty \int_0^t \int_0^\infty |K_x(x, y, t-\tau)| dy d\tau \\
 &\leq |p|_\infty |g|_\infty C_2(0) \sqrt{t}
 \end{aligned} \tag{3.4.36}$$

and also

$$\begin{aligned}
 \tilde{v}_{xx} &= \phi_{xx} + \int_0^t \int_0^\infty K_{xx}(x, y, t-\tau) \left[ -p(w(y, \tau)) g(\tilde{v}(y, \tau)) \right. \\
 &\quad \left. + p(w(x, \tau)) g(\tilde{v}(y, \tau)) \right] dy d\tau \\
 \Rightarrow |\tilde{v}_{xx} - \phi_{xx}| &\leq \left[ |g|_\infty |p|_\alpha (|w_x|_\infty)^\alpha + |p|_\infty |g|_\alpha (|\tilde{v}_x|_\infty)^\alpha \right] \times \\
 &\quad \int_0^t \int_0^\infty |K_{xx}(x, y, t-\tau)| |y - x|^\alpha dy d\tau \\
 &\leq \left[ |g|_\infty |p|_\alpha (|w_x|_\infty)^\alpha + |p|_\infty |g|_\alpha (|\tilde{v}_x|_\infty)^\alpha \right] \frac{C_3(\alpha)}{\alpha} t^{\alpha/2}
 \end{aligned} \tag{3.4.37}$$

Now substituting the bound for  $|w_x|$  and  $|\tilde{v}_x|$ , (3.4.34) and (3.4.37) will become respectively

$$\begin{aligned}
 |w_{xx} - \psi_{xx}| &\leq \left[ |g|_\infty |p|_\alpha \left\{ |\psi_x| + C_2(0) \sqrt{t} |p|_\infty |g|_\infty \right\}^\alpha \right. \\
 &\quad \left. + |p|_\infty |g|_\alpha \left\{ |\phi_x| + |p|_\infty |g|_\infty C_2(0) \sqrt{t} \right\}^\alpha \right] \frac{C_3(\alpha)}{\alpha} t^{\alpha/2}
 \end{aligned} \tag{3.4.38}$$

and

$$|\tilde{v}_{xx} - \phi_{xx}| \leq \left[ |g|_{\infty} |p|_{\alpha} \left\{ |\psi_x|_{\infty} + C_2(0) \sqrt{t} |p|_{\infty} |g|_{\infty} \right\}^{\alpha} + |p|_{\infty} |g|_{\alpha} \left\{ |\phi_x|_{\infty} + C_2(0) \sqrt{t} |p|_{\infty} |g|_{\infty} \right\}^{\alpha} \right] \frac{C_3(\alpha)}{\alpha} t^{\alpha/2} \quad (3.4.39)$$

From these two inequalities (3.4.38) and (3.4.39), we can conclude that

$$w_{xx} - \psi_{xx} \longrightarrow 0 \quad \text{and} \quad \tilde{v}_{xx} - \phi_{xx} \longrightarrow 0, \text{ like } O(t^{\alpha/2})$$

if  $\|p\|_{\alpha} \leq E$  and  $\|g\|_{\alpha} \leq E$  (3.4.40)

The last part  $(|v_t|_{\infty})^{\alpha}$  which is left, has been calculated in chapter II .

Now we define the sequence of iterates  $\{p^{(n)}\}$  defined by

$$p^{(n+1)}(t) = T_{\theta} [p^{(n)}] (t)$$

since  $f$  is a fixed point of  $T_{\theta}$  , clearly by the estimate

$$\|f - p^{(n+1)}\|_{\alpha} \leq \Lambda_{37} \|f - p^{(n)}\|_{\alpha}$$

and consequently  $p^{(n)} - f \longrightarrow 0$  as  $n \longrightarrow \infty$ . Since  $C^{\alpha}$  is complete, so  $p^{(n)} \longrightarrow f$  in  $C^{\alpha}$ . To see that  $T_{\theta}$  has a unique fixed point, we proceed by the method of contradiction. Suppose  $h$  is another fixed point of  $T_{\theta}$  , but we know that

$$\|h - f\|_{\alpha} = \|T_{\theta}[h] - T_{\theta}[f]\|_{\alpha} \leq \Lambda_{37} \|h - f\|_{\alpha} < \|h - f\|_{\alpha}$$

This is possible if  $f = h$ .

This completes basic assertions of this chapter.

### 3.5 : CONCLUSIONS :

In this chapter we have shown the uniqueness of a solution which is established by the method of contraction. A large number of estimates have

been divided to arrive at the results. The constants are derived explicitly. With the assumptions on flat initial values the local contractively result is true for small time interval. The present work is possibly the first work for a coupled system of two equations along these lines.

There are several directions in which the work remains to supplemented. First of all we have assumed the Neumann boundary condition as the identity property is not satisfied for Dirichlet boundary conditions. So this theory will not be applicable to Dirichlet Boundary Value problem. Hence, possibly a different approach needs to be looked into. Recently HOLLIS and MORGAN [60], [62] have studied the wellposedness of the direct problem for system of R-D equations ( $n \geq 2$ ) considering mixed boundary conditions on a global boundary. Therefore, the inverse problems for more than two equations with mixed boundary condition on local boundary ( $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$ ) data possibly be studied later.

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## CHAPTER IV

# EXISTENCE OF A SOLUTION OF THE SOURCE PROBLEM IN A HEAT CONDUCTION EQUATION

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### 4.1 INTRODUCTION :

In this chapter we have considered the existence of an unknown forcing function as a function not only of the unknown solution but also of the spatial variable  $x$ . Assuming variation to be space variable, the form of the function is assumed to be  $f_1(u) + x f_2(u)$ . In view of the two independent functions entering into the unknown right hand side, boundary conditions are assumed at both the boundary points. Earlier PILANT and RUNDELL (1986,1988) [102], [104] have studied the problem in a semi-infinite domain considering right hand side as a function of "u" only.

The preview of this chapter is as follows. In section 2 the inverse problem is formulated. In section 3, an iteration scheme is outlined. An equivalence relation is established in section 4. Section 5 deals with the existence of a fixed point of a nonlinear map and a brief conclusion is given in section 6.

## 4.2 FORMULATION OF THE INVERSE PROBLEM :

We consider the following Heat equation in a spatially bounded domain,

$$u_t(x,t) - u_{xx}(x,t) = (1-x) f_1(u(x,t)) + x f_2(u(x,t)) + \gamma(x,t) ;$$

$$0 < x < 1 , t > 0 ; \quad (4.2.1)$$

$$u_x(0,t) = g_0(t) ; t > 0 ;$$

$$u_x(1,t) = g_1(t) ; t > 0 ; \quad (4.2.2)$$

$$u(x,0) = u_0(x) ; 0 \leq x \leq 1 ; \quad (4.2.3)$$

Where,  $\gamma$  is a known function and the functions  $f_1(.)$  and  $f_2(.)$  are unknown and are to be determined with the help of overposed boundary values of  $u$  at  $x = 0$  and  $x = 1$ , given in the following form.

$$u(0,t) = \theta(t) ; t > 0 ; \quad (4.2.4)$$

$$u(1,t) = q(t) ; t > 0 ; \quad (4.2.5)$$

An introduction of a comparison function  $\psi$ , defined below will reduce the complexity of computations.

$$\psi_t(x,t) - \psi_{xx}(x,t) = \gamma(x,t) ; 0 < x < 1 , t > 0 ; \quad (4.2.6)$$

$$\psi_x(0,t) = g_0(t) ; t > 0 ;$$

$$\psi_x(1,t) = g_1(t) ; t > 0 ; \quad (4.2.7)$$

$$\psi(x,0) = u_0(x) ; 0 \leq x \leq 1 ; \quad (4.2.8)$$

If one substitutes the values of  $u$  at  $x = 0$  and  $x = 1$  in the equation (4.2.1), then the following two equations result :

$$\theta'(t) - u_{xx}(0,t;f_1,f_2) = f_1(\theta(t)) + \gamma(0,t) \quad (4.2.9)$$

$$q'(t) - u_{xx}(1,t;f_1,f_2) = f_2(q(t)) + \gamma(1,t) \quad (4.2.10)$$

The above two equations provide the motivation for an iteration scheme to determine the functions  $f_1$  and  $f_2$ .

### 4.3 ITERATION SCHEME :

In order to find the solution of the inverse problem (4.2.1)-(4.2.5) the following iteration scheme is developed. For any given initial choice  $(f_1^{(0)}, f_2^{(0)})$  of  $(f_1, f_2)$  the corresponding direct problem is solved. Let  $u(x, t; f_1^{(0)}, f_2^{(0)})$  be the solution of

$$u_t(x, t) - u_{xx}(x, t; f_1^{(0)}, f_2^{(0)}) = (1 - x) f_1^{(0)}(u(x, t)) \\ + x f_2^{(0)}(u(x, t)) + \gamma(x, t); \quad x \in (0, 1), \quad t > 0;$$

$$u_x(x, t) = g_0(t), \quad u_x(1, t) = g_1(t); \quad t > 0; \quad (4.3.1)$$

$$u(x, 0) = u_0(x); \quad x \in [0, 1];$$

However, in general  $u(x, t; f_1^{(0)}, f_2^{(0)})$  will not satisfy the Dirichlet Boundary data at  $x = 0$  and  $x = 1$ , i.e.,

$$u(0, t; f_1^{(0)}, f_2^{(0)}) \neq \theta(t) \quad \text{and} \quad u(1, t; f_1^{(0)}, f_2^{(0)}) \neq q(t)$$

Therefore, one needs a strategy to update  $(f_1^{(0)}, f_2^{(0)})$ . Let  $(f_1^{(1)}, f_2^{(1)})$  be the new value generated by  $(f_1^{(0)}, f_2^{(0)})$  via the following updating scheme.

$$f_1^{(1)}(\theta(t)) = \theta'(t) - u_{xx}(0, t; f_1^{(0)}, f_2^{(0)}) - \gamma(0, t) \quad (4.3.2)$$

and

$$f_2^{(1)}(q(t)) = q'(t) - u_{xx}(1, t; f_1^{(0)}, f_2^{(0)}) - \gamma(1, t) \quad (4.3.3)$$

Proceeding along these lines, one has the following for  $k \in \mathbb{N}$ .

$$f_1^{(k+1)}(\theta(t)) = \theta'(t) - u_{xx}(0, t; f_1^{(k)}, f_2^{(k)}) - \gamma(0, t) \quad (4.3.4)$$

$$f_2^{(k+1)}(q(t)) = q'(t) - u_{xx}(1, t; f_1^{(k)}, f_2^{(k)}) - \gamma(1, t) \quad (4.3.5)$$

Before proceeding further, we introduce the following notion :

$$T_{\theta,q} [f_1, f_2](t) = \left[ \begin{aligned} &\theta'(t) - u_{xx}(0, t; f_1, f_2) - \gamma(0, t), \\ &q'(t) - u_{xx}(1, t; f_1, f_2) - \gamma(1, t) \end{aligned} \right] \quad (4.3.6)$$

Therefore, the iteration scheme can be restated in terms of  $T_{\theta,q}$  operator as

$$\begin{aligned} \left[ f_1^{(k+1)}(\theta(t)), f_2^{(k+1)}(q(t)) \right] &= T_{\theta,q} \left[ f_1^{(k)}, f_2^{(k)} \right](t) \\ &= \left[ \begin{aligned} &\theta'(t) - u_{xx}(0, t; f_1^{(k)}, f_2^{(k)}) - \gamma(0, t), \\ &q'(t) - u_{xx}(1, t; f_1^{(k)}, f_2^{(k)}) - \gamma(1, t) \end{aligned} \right] \end{aligned} \quad (4.3.7)$$

which on adding and subtracting  $f_j^{(k)}(.)$  for  $j = 1, 2$  and replacing  $\gamma - u_{xx}$  by its expression in the equation (4.3.1) becomes

$$\begin{aligned} &= \left[ f_1^{(k)}(\theta(t)) + \left\{ f_1^{(k)}(u^{(k)}(0, t; f_1, f_2)) - f_1^{(k)}(\theta(t)) \right. \right. \\ &\quad \left. \left. + \theta'(t) - u_t^{(k)}(0, t; f_1^{(k)}, f_2^{(k)}) \right\}, \right. \\ &\quad \left. f_2^{(k)}(q(t)) + \left\{ f_2^{(k)}(u^{(k)}(1, t; f_1, f_2)) - f_2^{(k)}(q(t)) \right. \right. \\ &\quad \left. \left. + q'(t) - u_t^{(k)}(1, t; f_1^{(k)}, f_2^{(k)}) \right\} \right] \\ &= \left[ f_1^{(k)}(\theta(t)) + F_1 \left\{ \theta(t) - u^{(k)}(0, t) \right\}, \right. \\ &\quad \left. f_2^{(k)}(q(t)) + F_2 \left\{ q(t) - u^{(k)}(1, t) \right\} \right] \end{aligned}$$

Here  $F_1$  and  $F_2$  are in general nonlinear functions.

#### 4.4. EQUIVALENCE RELATION :

The following lemma shows the equivalence of the solvability of the inverse problem to the existence of a solution of a fixed point problem.

DEFINITION : A function  $(f_1, f_2)$  will be called  $(\theta, q)$ -fixed point of  $T_{\theta, q}$  if given  $\theta = \theta(t)$  and  $q = q(t)$ ,

$$\begin{bmatrix} f_1(\theta(t)) , f_2(q(t)) \end{bmatrix} = T_{\theta, q} [f_1, f_2](t) \quad (4.4.1)$$

If  $f_1$  and  $f_2$  are uniformly Lipschitz, then the existence of  $(\theta, q)$ -fixed point of the operator  $T_{\theta, q}$  is equivalent to the solution of the inverse problem of determining the source function with the help of the superposed Dirichlet data. We state it as a lemma.

LEMMA 1 : If  $f_1$  and  $f_2$  are Lipschitz function, then  $\{u, (f_1, f_2)\}$  is a solution of (4.2.1)-(4.2.5) if and only if  $(f_1, f_2)$  is a  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ .

PROOF : Suppose  $\{u, (f_1, f_2)\}$  is a solution of (4.2.1)-(4.2.5). Now we show that  $(f_1, f_2)$  is a  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ . Since  $(u, f_1, f_2)$  is a solution triplet, it satisfies the following relation :

$$\begin{aligned} \begin{bmatrix} f_1(\theta(t)) , f_2(q(t)) \end{bmatrix} &= \begin{bmatrix} f_1(u(0, t)) , f_2(u(1, t)) \end{bmatrix} \\ &= \begin{bmatrix} u_t(0, t) - u_{xx}(0, t; f_1, f_2) - \gamma(0, t), \\ u_t(1, t) - u_{xx}(1, t; f_1, f_2) - \gamma(1, t) \end{bmatrix} \\ &= \begin{bmatrix} \theta'(t) - u_{xx}(0, t; f_1, f_2) - \gamma(0, t), \\ q'(t) - u_{xx}(1, t; f_1, f_2) - \gamma(1, t) \end{bmatrix} \\ &= T_{\theta, q} \begin{bmatrix} f_1, f_2 \end{bmatrix} (t) \end{aligned}$$

So  $(f_1, f_2)$  is a  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ .

Conversely, assuming  $(f_1, f_2)$  is a  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ , we claim that  $\{u, (f_1, f_2)\}$  is a solution of the inverse source problem. It is sufficient to show  $u(0, t) = \theta(t)$  and  $u(1, t) = q(t)$  for the I. B. V. problem (4.2.1)-(4.2.3) with  $f_1$  and  $f_2$  as the source functions.

Since  $\{u, (f_1, f_2)\}$  satisfies the differential equation (4.2.1)-(4.2.3) we have

$$\gamma(0, t) + u_{xx}(0, t; f_1, f_2) = u_t(0, t) - f_1(u(0, t)) \quad (4.4.2)$$

at  $x = 0$ , and

$$\gamma(1, t) + u_{xx}(1, t; f_1, f_2) = u_t(1, t) - f_2(u(1, t)) \quad (4.4.3)$$

at  $x = 1$ .

If  $(f_1, f_2)$  is  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ , then from the equations (4.4.2) and (4.4.3) we have,

$$\begin{aligned} \begin{bmatrix} f_1(\theta(t)) & f_2(q(t)) \end{bmatrix} &= T_{\theta, q} \begin{bmatrix} f_1 & f_2 \end{bmatrix} (t) \\ &= \begin{bmatrix} \theta'(t) - \left\{ u_{xx}(0, t; f_1, f_2) + \gamma(0, t) \right\} , \\ q'(t) - \left\{ u_{xx}(1, t; f_1, f_2) + \gamma(1, t) \right\} \end{bmatrix} \\ &= \begin{bmatrix} \theta'(t) - u_t(0, t; f_1, f_2) + f_1(u(0, t)) , \\ q'(t) - u_t(1, t; f_1, f_2) + f_2(u(1, t)) \end{bmatrix} \end{aligned}$$

Here  $\gamma + u_{xx}$  has been replaced by its equivalent expression from the equation (4.3.1). Thus the following holds.

$$\begin{aligned} &\begin{bmatrix} \theta'(t) - u_t(0, t; f_1, f_2) & q'(t) - u_t(1, t; f_1, f_2) \end{bmatrix} \\ &= \begin{bmatrix} f_1(\theta(t)) - f_1(u(0, t)) & f_2(q(t)) - f_2(u(1, t)) \end{bmatrix} \end{aligned} \quad (4.4.4)$$

Denoting  $\alpha$  and  $\beta$  by the expression below,

$$\alpha(t) = \theta(t) - u(0,t)$$

and

$$\beta(t) = q(t) - u(1,t)$$

and noting that  $f_1, f_2$  are assumed to be Lipschitzian with the Lipschitz constants  $D$  and  $C$  respectively, we get

$$\begin{aligned} |\alpha'(t)| &= |\theta'(t) - u_t(0,t)| \\ &= |f_1(\theta(t)) - f_1(u(0,t))| && \text{(from (4.4.4))} \\ &\leq D |\theta(t) - u(0,t)| \\ &\leq D |\alpha(t)| \end{aligned} \tag{4.4.5}$$

and similarly,

$$|\beta'(t)| \leq C |\beta(t)| \tag{4.4.6}$$

Since the overposed data  $\theta(t)$  and  $q(t)$  must agree with the initial data  $u_0(x)$  at  $x = 0$  and  $x = 1$  respectively, we have

$$\alpha(0) = \theta(0) - u(0,0) = 0 \quad \text{and} \quad \beta(0) = q(0) - u(1,0) = 0$$

Then, by Gronwall's lemma  $\alpha(t) = \beta(t) \equiv 0$  for  $t > 0$ . i.e, the overposed boundary values are also satisfied. ■

So the existence of a  $(\theta, q)$ -fixed point of  $T_{\theta, q}$ , ensures that there is a solution for the inverse problem under consideration.

Thus the original problem reduces in establishing the existence of a  $(\theta, q)$ -fixed point of the  $T_{\theta, q}$  operator.

#### 4.5 EXISTENCE OF A FIXED POINT OF $T_{\theta,q}$ :

First of all we turn our attention to the following preliminary considerations. Let  $K$  be the Neumann function for the bounded region  $(0 \leq x \leq 1)$  with homogeneous Neumann data. Then  $K$  has the following form

$$K = K(x, y, t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} + e^{-(x+y-2n)^2/4t} \right] \quad (4.5.1)$$

If  $v$  denotes the difference between  $u$  and  $\psi$  given in (4.2.1) and (4.2.6) respectively then,  $v$  satisfies the following equation :

$$v_t(x, t) - v_{xx}(x, t) = (1 - x) f_1(u) + x f_2(u), \quad x \in (0, 1), \quad t > 0; \quad (4.5.2a)$$

$$v_x(0, t) = 0, \quad v_x(1, t) = 0; \quad t > 0; \quad (4.5.2b)$$

$$v(x, 0) = 0; \quad x > 0; \quad (4.5.2c)$$

Therefore, the solution of (4.5.2) can be written as

$$v = u - \psi = \int_0^t \int_0^1 K(x, y, t-\tau) \left[ (1-y) f_1(u(y, \tau)) + y f_2(u(y, \tau)) \right] dy d\tau \quad (4.5.3)$$

Now differentiating  $v$  twice with respect to  $x$ , one gets

$$\begin{aligned} v_{xx} &= u_{xx} - \psi_{xx} = \int_0^t \int_0^1 K_{xx}(x, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\ &+ \int_0^t \int_0^1 K_{xx}(x, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \end{aligned} \quad (4.5.4)$$

Hence the second derivative of  $u$  at  $x = 0$  and  $x = 1$  are given respectively by

$$\begin{aligned} u_{xx}(0, t; f_1, f_2) &= \psi_{xx}(0, t) + \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\ &+ \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \end{aligned} \quad (4.5.5)$$

and

$$\begin{aligned} u_{xx}(1, t; f_1, f_2) &= \psi_{xx}(1, t) + \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\ &+ \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \end{aligned} \quad (4.5.6)$$

From the definition of  $T_{\theta, q}$ , we have,

$$\begin{aligned} T_{\theta, q}[f_1, f_2](t) &= \left[ \theta'(t) - u_{xx}(0, t; f_1, f_2) - \gamma(0, t), \right. \\ &\quad \left. q'(t) - u_{xx}(1, t; f_1, f_2) - \gamma(1, t) \right] \end{aligned} \quad (4.5.7)$$

Now replacing  $\gamma$  by  $\psi_t - \psi_{xx}$  and  $u_{xx} - \psi_{xx}$  by their expressions in the equations (4.5.5) and (4.5.6), the equation (4.5.7), reduces to the form

$$\begin{aligned} T_{\theta, q}[f_1, f_2](t) &= \left[ \theta'(t) - u_{xx}(0, t; f_1, f_2) - \gamma(0, t), \right. \\ &\quad \left. q'(t) - u_{xx}(1, t; f_1, f_2) - \gamma(1, t) \right] \\ &= \left[ \theta'(t) - u_{xx}(0, t; f_1, f_2) + \psi_{xx}(0, t) - \psi_t(0, t), \right. \\ &\quad \left. q'(t) - u_{xx}(1, t; f_1, f_2) + \psi_{xx}(1, t) - \psi_t(1, t) \right] \\ &= \left[ \theta'(t) - \psi_t(0, t) - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) f_1(u(y, \tau)) dy d\tau \right. \\ &\quad \left. - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau, \right. \\ &\quad \left. q'(t) - \psi_t(1, t) - \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) f_1(u(y, \tau)) dy d\tau \right. \\ &\quad \left. - \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \right] \\ &= \left[ \theta'(t) - \psi_t(0, t) - K_0(f_1, f_2), q'(t) - \psi_t(1, t) - K_1(f_1, f_2) \right] \end{aligned} \quad (4.5.8)$$

We need the following result for subsequent use and express it as a lemma.

LEMMA 2 : The Neumann function  $K$  defined in (4.5.1) satisfies

$$\int_0^1 K_{xx}(x,y,t) dy = \int_0^1 K_x(x,y,t) dy = 0 \quad \text{for all } x \in [0,1] \quad (4.5.9)$$

PROOF : Indeed,

$$K(x,y,t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} + e^{-(x+y-2n)^2/4t} \right]$$

so, differentiating  $K$ , with respect to  $x$  gives

$$K_x(x,y,t) = \sum_{n=-\infty}^{\infty} - \frac{1}{\sqrt{4\pi t}} \left[ \frac{(x-y-2n)}{2t} e^{-(x-y-2n)^2/4t} + \frac{(x+y-2n)}{2t} e^{-(x+y-2n)^2/4t} \right] \quad (4.5.10)$$

Another differentiation of  $K_x$  with reference to  $x$ , yields

$$\begin{aligned} K_{xx}(x,y,t) &= \sum_{n=-\infty}^{\infty} - \frac{1}{\sqrt{4\pi t}} \left[ - \frac{(x-y-2n)^2}{4t^2} e^{-(x-y-2n)^2/4t} + \frac{1}{2t} e^{-(x-y-2n)^2/4t} - \frac{(x+y-2n)^2}{4t^2} e^{-(x+y-2n)^2/4t} + \frac{1}{2t} e^{-(x+y-2n)^2/4t} \right] \\ &= \sum_{n=-\infty}^{\infty} - \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x-y-2n)^2}{4t^2} \right) + e^{-(x+y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x+y-2n)^2}{4t^2} \right) \right] \end{aligned} \quad (4.5.11)$$

In order to prove (4.5.9), we proceed step wise. Integration by parts plays a major role in establishing this equality. Inserting the expression in (4.5.11) for  $K_{xx}$ , we get

$$\begin{aligned}
& \int_0^1 K_{xx}(x, y, t) dy \\
&= - \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x-y-2n)^2}{4t^2} \right) \right. \\
&\quad \left. + e^{-(x+y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x+y-2n)^2}{4t^2} \right) \right] dy
\end{aligned}$$

or,

$$\begin{aligned}
\int_0^1 K_{xx}(x, y, t) dy &= \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y-2n)^2/4t} \frac{(x-y-2n)^2}{4t^2} dy \\
&\quad + \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x+y-2n)^2/4t} \frac{(x+y-2n)^2}{4t^2} dy \\
&\quad - \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{2t\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} + e^{-(x+y-2n)^2/4t} \right] dy
\end{aligned} \tag{4.5.12}$$

Now,

$$\begin{aligned}
& \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y-2n)^2/4t} \frac{(x-y-2n)^2}{4t^2} dy \\
&= \int_0^1 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y-2n)^2/4t} \frac{(x-y-2n)}{2t} dy \times \frac{(x-y-2n)}{2t} dy \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left\{ \frac{(x-y-2n)}{2t} e^{-(x-y-2n)^2/4t} \right\}_0^1 \\
&\quad + \int_0^1 \frac{1}{2t} e^{-(x-y-2n)^2/4t} dy \Bigg\} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ \frac{(x-y-2n)}{2t} e^{-(x-y-2n)^2/4t} \right]_0^1 \\
&\quad + \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \int_0^1 \frac{1}{2t} e^{-(x-y-2n)^2/4t} dy
\end{aligned} \tag{4.5.13}$$

Similarly, integrating the second term on the R. H. S of expressions (4.5.12) by parts we have

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ -\frac{(x+y-2n)}{2t} \times e^{-(x+y-2n)^2/4t} \right]_0^1 \\ & + \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \int_0^1 \frac{1}{2t} e^{-(x+y-2n)^2/4t} dy \end{aligned} \quad (4.5.14)$$

Now substituting the respective terms of equation (4.5.12) by expressions (4.5.13) and (4.5.14) we get

$$\begin{aligned} \int_0^1 K_{xx}(x,y,t) dy &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left\{ \frac{(x-1-2n)}{2t} e^{-(x-1-2n)^2/4t} \right. \\ &\quad \left. - \frac{(x+1-2n)}{2t} e^{-(x+1-2n)^2/4t} \right\} \\ &= 0 \end{aligned} \quad (4.5.15)$$

Similarly, integrating by parts one can show  $\int_0^1 K_x(x,y,t) dy = 0$  and hence the lemma is proved. ■

**COROLLARY 1 :** If  $h$  is a continuous function, then

$$\begin{aligned} \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) h(x,\tau) dy d\tau &= \int_0^t h(x,\tau) \left[ \int_0^1 K_{xx}(x,y,t-\tau) dy \right] d\tau \\ &= 0 \end{aligned} \quad (4.5.16)$$

We record the following statement for later use in the form of a corollary :

**COROLLARY 2 :** If for  $h \in C^\alpha$ ,  $\alpha > 0$ , then

$$\begin{aligned} & \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) h(x,\tau) dy d\tau \\ &= \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) \left[ h(y,\tau) - h(x,\tau) \right] dy d\tau \end{aligned} \quad (4.5.17)$$

In order to prove the main theorem of this section we record the following three assumptions :

$$(B1) \quad u_0 \in C^{2+\alpha}[0,1], \quad u_x(0,t) = 0, \quad u_x(1,t) = 0$$

(B2) The functions  $\theta(t)$  and  $q(t)$  are monotone functions whose derivatives lie in  $C^\alpha[0,1]$  and  $\inf_{t \geq 0} |\theta'(t)| \geq \delta_1 > 0$ ,  $\inf_{t \geq 0} |q'(t)| \geq \delta_2 > 0$  for some  $\delta_1$  and  $\delta_2$ .

$$(B3) \quad f_1 \text{ and } f_2 \in B_E \equiv \left\{ \cdot \mid \|\cdot\|_\alpha \leq E \right\} \text{ for some constant } E.$$

Now we state the following basic theorem :

**THEOREM 1 :** If the assumptions (B1)-(B3) are true, then for sufficiently small  $T$  and  $0 < \alpha < 1/2$ ,  $T_{\theta,q}$  maps a ball in maps  $C^\alpha \times C^\alpha$  into itself.

**PROOF :**

Given  $f_1$  and  $f_2$  belonging to  $C^\alpha$ , we define the functions  $\tilde{f}_1$  and  $\tilde{f}_2$  by

$$\left[ \tilde{f}_1(\theta(t)), \tilde{f}_2(q(t)) \right] = T_{\theta,q} [f_1, f_2](t) \quad (4.5.18)$$

so

$$\begin{aligned} & \left[ \tilde{f}_1(\theta(t)), \tilde{f}_2(q(t)) \right] \\ &= \left[ \theta'(t) - \psi_t(0,t) - \int_0^t \int_0^1 K_{xx}(0,y,t-\tau) f_1(u(y,\tau)) dy d\tau \right. \\ & \quad \left. - \int_0^t \int_0^1 K_{xx}(0,y,t-\tau) y \left[ f_2(u(y,\tau)) - f_1(u(y,\tau)) \right] dy d\tau, \right. \\ & \quad \left. q'(t) - \psi_t(1,t) - \int_0^t \int_0^1 K_{xx}(1,y,t-\tau) f_1(u(y,\tau)) dy d\tau \right. \\ & \quad \left. - \int_0^t \int_0^1 K_{xx}(1,y,t-\tau) y \left[ f_2(u(y,\tau)) - f_1(u(y,\tau)) \right] dy d\tau \right] \quad (4.5.19) \end{aligned}$$

First of all we estimate the third and fourth terms in the expression for  $\tilde{f}_1$ .

Assume

$$\begin{aligned} I_1 &= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\ &= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ f_1(u(y, \tau)) - f_1(u(0, \tau)) \right] dy d\tau \end{aligned}$$

(at  $x = 0$ , the identity (4.5.17) gives the above result.)

Using the fact that  $f_1 \in C^\alpha$  and expanding  $u(y, \tau)$  by Taylor's expansion about  $(0, \tau)$ , we obtain

$$|I_1| \leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |f_1|_\alpha |u_x(0, \tau) y + u_{xx}(\theta, \tau) (y^2/2)|^\alpha dy d\tau$$

By assumption B1,  $u_x(0, t)$  is zero. So

$$\begin{aligned} |I_1| &\leq |f_1|_\alpha (|u_{xx}|_\infty)^\alpha \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| (y^2/2)^\alpha dy d\tau \\ &\leq 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha \frac{1}{\alpha} C_3(0, 0, 2\alpha) t^\alpha \end{aligned} \quad (4.5.20)$$

(By (B3) of Appendix B).

we can not apply the identity property for estimation of the last term of  $\tilde{f}_1(\theta(t))$ , because  $y$  is multiplied to  $K_{xx}$ . However, it can be estimated directly as

$$|I_2| \leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |y| \left[ |f_2|_\infty + |f_1|_\infty \right] dy d\tau$$

$$\text{or,} \quad |I_2| \leq \left( |f_1|_\infty + |f_2|_\infty \right) C_3(0, 0, 1) t^{1/2} \quad (4.5.21)$$

similarly, the estimate for  $\tilde{f}_2$  is evaluated. Let us denote

$$\begin{aligned}
I_3 &= \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\
&= \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) \left[ f_1(u(y, \tau)) - f_1(u(1, \tau)) \right] dy d\tau
\end{aligned}$$

$u(y, \tau)$  can be expanded about  $y = 1$  and we get

$$\begin{aligned}
|I_3| &\leq \int_0^t \int_0^1 |K_{xx}(1, y, t-\tau)| |f_1|_\alpha |u_x(1, \tau)| (y-1) \\
&\quad + u_{xx}(\theta, \tau) ((y-1)^2/2)^\alpha dy d\tau
\end{aligned}$$

Since by assumption B1,  $u_x(1, \tau) = 0$ . Therefore we have,

$$\begin{aligned}
|I_3| &\leq (|u_{xx}|_\infty)^\alpha |f_1|_\alpha \int_0^t \int_0^1 |K_{xx}(1, y, t-\tau)| \frac{|y-1|^{2\alpha}}{2^\alpha} dy d\tau \\
&\leq 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_3(1, 0, 2\alpha) \frac{t^\alpha}{\alpha}
\end{aligned} \tag{4.5.22}$$

The last term of  $\tilde{f}_2$  which we call  $I_4$  has an estimate analogous to (4.5.21) i.e., at  $x = 1$ , this is given by

$$|I_4| \leq \left( |f_2|_\infty + |f_1|_\infty \right) C_3(1, 0, 1) t^{1/2} \tag{4.5.23}$$

Hence, from the expressions (4.5.19) and the estimate (4.5.20), the first component of (4.5.19) can be estimated to get

$$\begin{aligned}
\|\tilde{f}_1\|_\infty &\leq \|\theta' - \psi_t\|_\infty + 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_3(0, 0, 2\alpha) t^\alpha \\
&\quad + \left( |f_1|_\infty + |f_2|_\infty \right) C_3(0, 0, 1) t^{1/2}
\end{aligned} \tag{4.5.24}$$

Similarly using (4.5.22) and (4.5.23), the second component of (4.5.19) is estimated to give

$$\begin{aligned} \|\tilde{f}_2\|_\infty \leq \|\theta' - \psi_t\|_\infty + 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_3(1,0,2\alpha) t^\alpha \\ + \left( |f_1|_\infty + |f_2|_\infty \right) C_3(1,0,1) t^{1/2} \end{aligned} \quad (4.5.25)$$

The  $\alpha$ -norm is defined as  $\|\cdot\|_\alpha = \|\cdot\|_\infty + |\cdot|_\alpha$ . Therefore, in order to calculate the  $\alpha$ -semi norm of  $\tilde{f}_1$  and  $\tilde{f}_2$ , one needs to estimate  $|\tilde{f}_1(\theta(t_1)) - \tilde{f}_1(\theta(t_2))|$  and  $|\tilde{f}_2(q(t_1)) - \tilde{f}_2(q(t_2))|$  for  $t_1 \neq t_2$  where  $\theta(t)$  and  $q(t)$  are monotone functions. Without loss of generality we assume  $t_1 > t_2$ .

The difference between  $\tilde{f}_1(\cdot)$  at two different time measurements is expressed as follows.

$$\begin{aligned} \tilde{f}_1(\theta(t_1)) - \tilde{f}_1(\theta(t_2)) &= \theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2) \\ &- \int_0^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) f_1(u(y, \tau)) dy d\tau \\ &+ \int_0^{t_2} \int_0^1 K_{xx}(0, y, t_2 - \tau) f_1(u(y, \tau)) dy d\tau \\ &- \int_0^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\ &+ \int_0^{t_2} \int_0^1 K_{xx}(0, y, t_2 - \tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \end{aligned} \quad (4.5.26)$$

Let  $I_5$  stand for the expression,

$$\begin{aligned} I_5 &= - \int_0^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) f_1(u(y, \tau)) dy d\tau \\ &+ \int_0^{t_2} \int_0^1 K_{xx}(0, y, t_2 - \tau) f_1(u(y, \tau)) dy d\tau \\ &= \int_0^{t_2} \int_0^1 \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] f_1(u(y, \tau)) dy d\tau \end{aligned}$$

$$\begin{aligned}
& + \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) f_1(u(y, \tau)) dy d\tau \\
& = I_{5,1} + I_{5,2}
\end{aligned} \tag{4.5.27}$$

Making use of the identity property at  $x = 0$ ,  $I_{5,2}$  is estimated as

$$\begin{aligned}
I_{5,2} &= \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) f_1(u(y, \tau)) dy d\tau \\
&= \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ f_1(u(y, \tau)) - f_1(u(0, \tau)) \right] dy d\tau
\end{aligned}$$

Taking modulus on both the sides and estimating the R. H. S., we have,

$$\begin{aligned}
|I_{5,2}| &\leq \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |f_1|_\alpha |u_{xx}(\theta, \tau) (y^2/2)|^\alpha dy d\tau \\
&\leq 2^{-\alpha} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t - \tau)| |y|^{2\alpha} dy d\tau \\
&\leq 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_3(0, 0, 2\alpha) \frac{|t_1 - t_2|^\alpha}{\alpha}
\end{aligned} \tag{4.5.28}$$

Now the use of mean value theorem modifies  $I_{5,1}$  into

$$I_{5,1} = \int_0^{t_2} \int_0^1 \int_{t_2}^{t_1} K_{xxt}(0, y, s - \tau) \left[ f_1(u(y, \tau)) - f_1(u(0, \tau)) \right] ds dy d\tau$$

so,

$$\begin{aligned}
|I_{5,1}| &\leq \int_0^{t_2} \int_0^1 \int_{t_2}^{t_1} |K_{xxt}(0, y, s - \tau)| |f_1(u(y, \tau)) - f_1(u(0, \tau))| ds dy d\tau \\
&\leq 2^{-\alpha} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s - \tau)| |y|^{2\alpha} dy d\tau ds \\
&\leq 2^{-\alpha} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_4(0, 0, 2\alpha) \int_{t_2}^{t_1} \int_0^{t_2} |s - \tau|^{\alpha-2} d\tau ds \quad (\text{Appendix B4})
\end{aligned}$$

$$\leq 2^{-\alpha} |f_1|_{\alpha} (|u_{xx}|_{\infty})^{\alpha} \frac{C_4(0,0,2\alpha)}{\alpha(1-\alpha)} |t_1 - t_2|^{\alpha} \quad (4.5.29)$$

Let  $I_6$  denote the last two terms of (4.5.26). The estimation technique for  $I_6$  is analogous to  $I_5$ . Indeed, it can be written equivalently as

$$\begin{aligned} I_6 &= \int_0^{t_2} \int_0^1 \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] y \times \\ &\quad \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\ &+ \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\ &= I_{6,1} + I_{6,2} \end{aligned} \quad (4.5.30)$$

$I_{6,2}$  is estimated as below

$$\begin{aligned} |I_{6,2}| &\leq \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |y| |f_2(u(y, \tau))| dy d\tau \\ &+ \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |y| |f_1(u(y, \tau))| dy d\tau \\ &\leq \left( |f_2|_{\infty} + |f_1|_{\infty} \right) C_3(0,0,1) |t_1 - t_2|^{1/2} \end{aligned} \quad (4.5.31)$$

$I_{6,1}$  can be estimated as follows. Taking modulus on both sides of

$$I_{6,1} = \int_0^{t_2} \int_0^1 \int_{t_2}^{t_1} K_{xxt}(0, y, s - \tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau$$

we have,

$$\begin{aligned} |I_{6,1}| &\leq \left( |f_1|_{\infty} + |f_2|_{\infty} \right) \frac{C_4(0,0,1)}{(1/2)(1-1/2)} |t_1 - t_2|^{1/2} \\ &\leq 4 \left( |f_1|_{\infty} + |f_2|_{\infty} \right) C_4(0,0,1) |t_1 - t_2|^{1/2} \end{aligned} \quad (4.5.32)$$

Collecting the estimates (4.5.28), (4.5.29), (4.5.31) and (4.5.32), we write

$$\begin{aligned}
 |\tilde{f}_1(\theta(t_1)) - \tilde{f}_1(\theta(t_2))| &\leq |\theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2)| \\
 &\quad + \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha |t_1 - t_2|^\alpha \\
 &\quad + C_b(0,0,1) \left( |f_1|_\infty + |f_2|_\infty \right) |t_1 - t_2|^{1/2} \quad (4.5.33)
 \end{aligned}$$

$$\text{where } C_a(0,0,\alpha) = 2^{-\alpha} C_4(0,0,2\alpha) + 2^{-(\alpha+1)} (1-\alpha) C_3(0,0,2\alpha) \quad (4.5.34)$$

$$\text{and } C_b(0,0,1) = 2^2 C_4(0,0,1) + C_3(0,0,1) \quad (4.5.35)$$

Hence, dividing throughout by  $|\theta(t_1) - \theta(t_2)|^\alpha$  to obtain an estimate for the  $\alpha$ -semi norm of  $\tilde{f}_1$ , we have

$$\begin{aligned}
 &\frac{|\tilde{f}_1(\theta(t_1)) - \tilde{f}_1(\theta(t_2))|}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 &\leq \frac{|\theta'(t_1) - \psi_t(0, t_1) - \theta'(t_2) + \psi_t(0, t_2)|}{|t_1 - t_2|^\alpha} \times \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 &\quad + \left\{ \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha \right. \\
 &\quad \left. + C_b(0,0,1) \left( |f_1|_\infty + |f_2|_\infty \right) |t_1 - t_2|^{(1/2)-\alpha} \right\} \times \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha}
 \end{aligned}$$

or,

$$\begin{aligned}
 |\tilde{f}_1|_\alpha &\leq \frac{|\theta' - \psi_t|_\alpha}{(\inf |\theta'|)^\alpha} + \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1|_\infty \frac{(|u_{xx}|_\infty)^\alpha}{(\inf |\theta'|)^\alpha} \\
 &\quad + \frac{C_b(0,0,1)}{(\inf |\theta'|)^\alpha} \left( |f_1|_\infty + |f_2|_\infty \right) |t_1 - t_2|^{(1/2)-\alpha} \quad (4.5.36)
 \end{aligned}$$

Towards the estimation of the  $\alpha$ -seminorm of  $\tilde{f}_2$ , we have from the equation (4.5.19)

$$\begin{aligned}\tilde{f}_2(q(t)) &= q'(t) - \psi_t(0, t) - \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) f_1(u(y, \tau)) dy d\tau \\ &\quad - \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) y \left[ f_2(u(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau\end{aligned}$$

An analogous procedure provides the following estimate for  $|\tilde{f}_2|_\alpha$

$$\begin{aligned}|\tilde{f}_2|_\alpha &\leq \frac{|q' - \psi_t|}{(\inf |q'|)^\alpha} + \frac{C_a(0, 0, \alpha)}{\alpha(1 - \alpha)} |f_1|_\alpha \left( \frac{|u_{xx}|_\infty}{\inf |q'|} \right)^\alpha \\ &\quad + \frac{C_b(1, 0, 1)}{(\inf |q'|)^\alpha} \left( |f_1|_\infty + |f_2|_\infty \right) |t_1 - t_2|^{(1/2)-\alpha}\end{aligned}\quad (4.5.37)$$

In view of (4.5.36) and (4.5.37) we need to take  $0 < \alpha < 1/2$ . Now to find the  $\alpha$ -norm of  $\tilde{f}_1$  we combine (4.5.24) and (4.5.36). Since (4.5.24) gives

$$\begin{aligned}\|\tilde{f}_1\|_\infty &= \|\theta' - \psi_t\|_\infty + 2^{-(\alpha+1)} |f_1|_\alpha (|u_{xx}|_\infty)^\alpha C_3(0, 0, 2\alpha) t^\alpha \\ &\quad + \left( |f_1|_\infty + |f_2|_\infty \right) C_3(0, 0, 1) t^{1/2}\end{aligned}$$

and  $\|\tilde{f}_1\|_\alpha = \|\tilde{f}_1\|_\infty + |\tilde{f}_1|_\alpha$ , we have

$$\begin{aligned}\|\tilde{f}_1\|_\alpha &\leq \max \left\{ 1, (\inf |\theta'|)^{-\alpha} \right\} \|\theta' - \psi_t\|_\alpha \\ &\quad + \left\{ \frac{C_a(0, 0, \alpha)}{\alpha(1 - \alpha)} \left[ 1 + t^\alpha (\inf |\theta'|)^\alpha \right] \left( \frac{|u_{xx}|_\infty}{\inf |\theta'|} \right)^\alpha \right\} |f_1|_\alpha \\ &\quad + \frac{C_b(0, 0, 1)}{(\inf |\theta'|)^\alpha} \left[ |t_1 - t_2|^{(1/2)-\alpha} + t^{1/2} (\inf |\theta'|)^\alpha \right] \left( |f_1|_\infty + |f_2|_\infty \right)\end{aligned}\quad (4.5.38)$$

Similar expression can also be given for  $\|\tilde{f}_2\|_\alpha$ .

Now, we study the behavior of the coefficient of  $\|f_1\|_\alpha$  in the above estimate, as  $t$  approaches zero and show that

$$\frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} \left[ 1 + t^\alpha (\inf |\theta'|)^\alpha \right] \left( \frac{|u_{xx}|_\infty}{\inf |\theta'|} \right)^\alpha \quad \text{and}$$

$$\frac{C_b(0,0,1)}{(\inf |\theta'|)^\alpha} \left[ |t_1 - t_2|^{(1/2)-\alpha} + t^\alpha (\inf |\theta'|)^\alpha \right] \quad \text{are sufficiently small}$$

for small values of " $t$ ".

Towards the end we compute a bound for  $\|u_{xx}\|_\infty$ . In view of the solution of nonhomogenous equation of (4.2.1)-(4.2.3), we have

$$\begin{aligned} u_{xx} - \psi_{xx} &= \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) f_1(u(y,\tau)) dy d\tau \\ &+ \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) y \left[ f_2(u(y,\tau)) - f_1(u(y,\tau)) \right] dy d\tau \end{aligned} \quad (4.5.39)$$

Due to the identity property, this will imply,

$$\begin{aligned} u_{xx} &= \psi_{xx} + \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) \left[ f_1(u(y,\tau)) - f_1(u(x,\tau)) \right] dy d\tau \\ &+ \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) \left[ y f_2(u(y,\tau)) - x f_2(u(x,\tau)) \right] dy d\tau \\ &- \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) \left[ y f_1(u(y,\tau)) - x f_1(u(x,\tau)) \right] dy d\tau \end{aligned} \quad (4.5.40)$$

Calculating the following part only

$$\begin{aligned} &y f_j(u(y,\tau)) - x f_j(u(x,\tau)) \text{ for } j = 1, 2. \\ &= (y - x) f_j(u(y,\tau)) + x (f_j(u(y,\tau)) - f_j(u(x,\tau))) \end{aligned} \quad (4.5.41)$$

Substituting (4.5.41) into equation (4.5.40) and taking the modulus, one gets

$$\begin{aligned}
 |u_{xx}| \leq & |\psi_{xx}| + \int_0^t \int_0^1 |K_{xx}(x,y,t-\tau)| |f_1|_\alpha |(y-x) u_x(\theta,\tau)|^\alpha dy d\tau \\
 & + \int_0^t \int_0^1 |K_{xx}(x,y,t-\tau)| |y-x| |f_2(u(y,\tau))| dy d\tau \\
 & + \int_0^t \int_0^1 |K_{xx}(x,y,t-\tau)| |x| |f_2|_\alpha |(y-x) u_x(\theta,\tau)|^\alpha dy d\tau \\
 & + \int_0^t \int_0^1 |K_{xx}(x,y,t-\tau)| |y-x| |f_1(u(y,\tau))| dy d\tau \\
 & + \int_0^t \int_0^1 |K_{xx}(x,y,t-\tau)| |x| |f_1|_\alpha |(y-x) u_x(\theta,\tau)|^\alpha dy d\tau
 \end{aligned}$$

So, this implies,

$$\begin{aligned}
 |u_{xx}| \leq & |\psi_{xx}| + \left( |f_1|_\alpha + |x| |f_1|_\alpha + |x| |f_2|_\alpha \right) C_3(x,0,\alpha) (|u_x|_\infty)^\alpha \frac{t^{\alpha/2}}{\alpha} \\
 & + \left( |f_2|_\infty + |f_1|_\infty \right) C_3(x,0,1) t^{1/2}
 \end{aligned} \tag{4.5.42}$$

Since, in the above estimate  $|u_x|_\infty$  is involved, next we obtain an estimate for  $|u_x|_\infty$  in an analogous manner.

$$\begin{aligned}
 u_x &= \psi_x + \int_0^t \int_0^1 K_x(x,y,t-\tau) f_1(u(y,\tau)) dy d\tau \\
 &+ \int_0^t \int_0^1 K_x(x,y,t-\tau) y \left[ f_2(u(y,\tau)) - f_1(u(y,\tau)) \right] dy d\tau \\
 &= \psi_x + \int_0^t \int_0^1 K_x(x,y,t-\tau) \left[ f_1(u(y,\tau)) - f_1(u(x,\tau)) \right] dy d\tau \\
 &+ \int_0^t \int_0^1 K_x(x,y,t-\tau) x \left[ f_2(u(y,\tau)) - f_2(u(x,\tau)) \right] dy d\tau \\
 &+ \int_0^t \int_0^1 K_x(x,y,t-\tau) (y-x) f_2(u(y,\tau)) dy d\tau \\
 &- \int_0^t \int_0^1 K_x(x,y,t-\tau) x \left[ f_1(u(y,\tau)) - f_1(u(x,\tau)) \right] dy d\tau
 \end{aligned}$$

$$- \int_0^t \int_0^1 K_x(x, y, t-\tau) (y-x) f_1(u(y, \tau)) dy d\tau$$

This implies from Appendix B (B4),

$$\begin{aligned} |u_x| \leq |\psi_x| + |f_1|_{\infty} C_2(x, 0, 0) t^{1/2} + |f_2|_{\infty} C_2(x, 0, 1) t \\ + |f_1|_{\infty} C_2(x, 0, 1) t + \left( |f_2|_{\infty} + |f_1|_{\infty} \right) |x| C_2(x, 0, 0) t^{1/2} \quad (4.5.43) \end{aligned}$$

Substituting (4.5.43) into (4.5.42), one has an estimate for  $|u_{xx}|_{\infty}$  explicitly. Therefore, for small values of "t" and for some constant E

$$|f_1|_{\alpha} + |f_1|_{\infty} \leq E \text{ and } |f_2|_{\alpha} + |f_2|_{\infty} \leq E$$

then  $u_{xx} \rightarrow \psi_{xx}$  like  $O(t^{\alpha/2})$ .

$$\text{Hence } \left( \frac{|u_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha} \cong \left( \frac{|\psi_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha} \cong \left( \frac{|u_0''(x)|}{\inf |\theta'(0)|} \right)^{\alpha}$$

So considering the flat initial value

$$\frac{C_a(0, 0, \alpha)}{\alpha(1-\alpha)} \left[ 1 + t^{\alpha} (\inf |\theta'|)^{\alpha} \right] \left( \frac{|u_{xx}|_{\infty}}{\inf |\theta'|} \right)^{\alpha} \text{ will become}$$

$$\frac{C_a(0, 0, \alpha)}{\alpha(1-\alpha)} \left( \frac{u_0''(x)}{\inf |\theta'(0)|} \right)^{\alpha}$$

$$\text{and } \frac{C_b(0, 0, 1)}{(\inf |\theta'|)^{\alpha}} \left[ |t_1 - t_2|^{(1/2)-\alpha} + t^{1/2} (\inf |\theta'|)^{\alpha} \right] \rightarrow 0$$

Therefore, (4.5.38) shows that

$$\|\tilde{f}_1\| \leq E \text{ and } \|\tilde{f}_2\| \leq E.$$

Hence

$$T_{\theta,q}[f_1, f_2](t) : B_E \subset C^\alpha[0, T] \times C^\alpha[0, T] \longrightarrow B_E \subset C^\alpha[0, T] \times C^\alpha[0, T] \quad \blacksquare$$

Finally, we note that  $T_{\theta,q}$  is a continuous map from  $C^\alpha \times C^\alpha$  into itself. This follows from the continuous dependence of the solution "u" of the direct problem on the right hand side function f.

LEMMA 2 : The set

$$V = \left\{ (f_1, f_2) : \|f_1\|_\infty \leq E, |f_1|_\alpha \leq E, \text{ for } i = 1, 2 \right\}$$

is closed in  $C^\beta \times C^\beta$  for  $0 < \beta < \alpha < 1$ .

PROOF : Clearly, V is a closed set in  $C^\alpha \times C^\alpha$ .

Let for  $i = 1, 2$ ,

$\{f_1^n\}$  be a sequence in V such that  $f_1^n \longrightarrow f_1$  in  $C^\beta \times C^\beta$  for  $\beta < \alpha$ .

We claim that  $f_1 \in V$ .

Since  $f_1^n \longrightarrow f_1$  in  $C^\beta \times C^\beta$

$$\Rightarrow \|f_1^n - f_1\|_\infty \longrightarrow 0$$

$$\text{So } \|f_1\|_\infty \leq \|f_1^n\|_\infty + \|f_1^n - f_1\|_\infty$$

$$\Rightarrow \|f_1\|_\infty \leq \overline{\lim} \left( \|f_1^n\|_\infty + \|f_1^n - f_1\|_\infty \right) \leq E.$$

Further , 
$$\frac{|f_1(x) - f_1(y)|}{|x - y|^\alpha} = \frac{|\lim (f_1^n(x) - f_1^n(y))|}{|x - y|^\alpha}$$

$$\leq \overline{\lim} \frac{|f_1^n(x) - f_1^n(y)|}{|x - y|^\alpha}$$

$$\leq \overline{\lim} E = E$$

$$\Rightarrow |f_1|_\alpha \leq E$$

Hence  $V$  is closed in  $C^\beta \times C^\beta$ . ■

**THEOREM 2 (EXISTENCE) :** Assume that  $T_{\theta,q}$  is a self mapping from  $V$  into itself, where  $V = \left\{ (f_1, f_2) : \|f_i\|_\infty \leq E, |f_i|_\alpha \leq E, \text{ for } i = 1, 2 \right\}$ . Then  $T_{\theta,q}$  has a fixed point.

**PROOF :** The set  $V$  is a closed, convex set in  $C^\alpha \times C^\alpha$ . By the above lemma  $V$  is also closed and convex set in  $C^\beta \times C^\beta$ . Since  $V$  is a bounded set in  $C^\alpha \times C^\alpha$ ,  $T_{\theta,q}V$  is precompact in  $C^\beta \times C^\beta$  (FRIEDMANN, P.188 [49]). Hence  $T_{\theta,q}$  is a completely continuous operator.

Therefore, considering  $T_{\theta,q} : V \subset C^\beta \times C^\beta \longrightarrow V \subset C^\beta \times C^\beta$ , and by SCHAUDER's fixed point theorem (FRIEDMANN, p. 189 [49]), we conclude that  $T_{\theta,q}$  has a fixed point in  $C^\beta \times C^\beta$ . ■

**REMARK :**

The existence of a limit of the iterated sequence  $\{f_i^{(k)}\}$  is not guaranteed by the existence theorem obtained above. In the next chapter we shall prove a centred contraction mapping Property of the  $T_{\theta,q}$  map under some additional conditions. This will be sufficient to show the convergence of the iterates  $\{f_i^{(k)}\}$  and hence establishes the convergence of the iterative scheme outlined in this chapter.

## 4.6 CONCLUSIONS :

The question of existence of the solution of a nonlinear source function problem of a fixed point of a Heat equation in a bounded spatial domain is answered in this chapter. The problem is shown to be equivalent to the existence of a fixed point of a certain nonlinear map. An iterative scheme is outlined via an operator what is shown to be self mapping on a bounded set in the Hölder space. A large number of domain estimates have been computed explicitly to arrive at this result. Subsequent use of the SCHAUDER's fixed point theorem ensures the existence of the original source problem.

## CHAPTER V

### UNIQUENESS OF THE SOLUTION OF THE HEAT CONDUCTION SOURCE

### PROBLEM AND CONVERGENCE OF AN ITERATIVE SCHEME

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#### 5.1 INTRODUCTION :

In this chapter the question of uniqueness and the convergence of iteration scheme for the inverse Heat conduction source problem is addressed. This is done by showing a centered contraction property for the map  $T_{\theta,q}$ .

**DEFINITION :** Let  $X$  be a Banach space and  $T$  is a map from  $U \subset X$  to  $V \subset X$ . Then  $T$  is called centered contraction map about  $u_0 \in U$  if

$$\|Tu_0 - Tu\| \leq \Lambda \|u_0 - u\|$$

with  $\forall u \in U$  and  $\Lambda$  is a constant independent of  $u$ , s. t.  $0 \leq \Lambda < 1$ .

Suppose that  $(f_1, f_2)$  is a  $(\theta, q)$ -fixed point of  $T_{\theta,q}$

$$\text{i. e., } \begin{bmatrix} f_1(\theta(t)), f_2(q(t)) \end{bmatrix} = T_{\theta,q} \begin{bmatrix} f_1, f_2 \end{bmatrix} (t) \quad (5.1.1)$$

and let  $(g_1, g_2)$  and  $(\tilde{g}_1, \tilde{g}_2)$  are related by

$$\begin{bmatrix} \tilde{g}_1(\theta(t)), \tilde{g}_2(q(t)) \end{bmatrix} = T_{\theta,q} \begin{bmatrix} g_1, g_2 \end{bmatrix} (t) \quad (5.1.2)$$

We show below that  $T_{\theta,q}$  is a centered contraction map in a neighborhood of the fixed point  $(f_1, f_2)$ . Therefore, we need to estimate  $f_1 - \tilde{g}_1$  and  $f_2 - \tilde{g}_2$  in terms of  $f_1 - g_1$  and  $f_2 - g_2$ , and obtain the following estimate :

$$\begin{aligned} \| (f_1, f_2) - (\tilde{g}_1, \tilde{g}_2) \|_{\alpha} &= \| T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2) \|_{\alpha} \\ &\leq \Lambda \| (f_1, f_2) - (g_1, g_2) \|_{\alpha} \end{aligned} \quad (5.1.3)$$

we note that

$$\begin{aligned} \| T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2) \|_{\alpha} &= \| (f_1, f_2) - (\tilde{g}_1, \tilde{g}_2) \|_{\alpha} \\ &= \left( \| f_1 - \tilde{g}_1 \|_{\alpha} + \| f_2 - \tilde{g}_2 \|_{\alpha} \right) \end{aligned} \quad (5.1.4)$$

where

$$\begin{aligned} \| T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2) \|_{\alpha} &= \| T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2) \|_{\infty} \\ &\quad + \| T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2) \|_{\alpha} \end{aligned} \quad (5.1.5)$$

In section 2 the main theorem is stated. The sup and the semi-norm for the operator are calculated in sections 3 and 4 respectively to prove the map to be contraction and hence the convergence of the iteration scheme obtained in the previous chapter. This chapter ends with a conclusion in section 5.

## 5.2 THE THEOREM :

In this section we state some additional hypotheses which need to be satisfied in order that the main theorem regarding centered contraction property can be proved.

Note that if  $f$  is a uniformly Lipschitz function and  $g \in C^{\alpha}$  then,  $f \circ g \in C^{\alpha}$  and

$$\| f \circ g \|_{\alpha} \leq \| f \|_{\infty} + \| f \|_1 \| g \|_{\alpha} \quad (5.2.1)$$

If  $f \circ g$  vanishes at some point then also

$$\| f \circ g \|_{\alpha} \leq C \| f \|_1 \| g \|_{\alpha} \leq C \| f \|_1 \| g \|_{\alpha} \quad (5.2.2)$$

for bounded domains  $f \in \text{Lip}$  class preserves  $C^{\alpha}$  regularity under composition. This is not sufficient to show uniqueness. Hence one extra property is needed.

**PROPERTY S :** A function  $f \in \text{Lip}$  class has the "property S" if given functions  $u, v \in C^{\alpha}$ , the mapping  $u \rightarrow f(u)$  is  $C^{\alpha}$ , that is there exists  $C_s < \infty$  and

$$\| f(u) - f(v) \|_{\alpha} \leq C_s \| u - v \|_{\alpha}. \quad (5.2.3)$$

Below it is assumed that  $f_1$  will satisfy the property S.

**THEOREM :** If the over posed-boundary value problem (4.2.1)-(4.2.5) defined in chapter IV possesses a  $(\theta, q)$ -fixed point  $(f_1, f_2)$  with property S, then it is unique. Furthermore, the operator  $T_{\theta, q}$  is a centered contraction on  $C^{\alpha} \times C^{\alpha}$  for  $\alpha < 1/2$  under the assumptions (B1)-(B3).

### 5.3 SUP-NORM ESTIMATION :

Let  $u(x, t; f_1, f_2)$  and  $v(x, t; g_1, g_2)$  be the respective solutions of the following partial differential equations.

$$u_t(x, t) - u_{xx}(x, t) = (1 - x)f_1(u(x, t)) + xf_2(u(x, t)) + \gamma(x, t); \quad (5.3.1a)$$

$$x \in (0, 1), t > 0;$$

$$u_x(0, t) = 0, \quad u_x(1, t) = 0; \quad t > 0; \quad (5.3.1b)$$

$$u(x, 0) = u_0(x); \quad x \in [0, 1]; \quad (5.3.1c)$$

$$u(0, t) = \theta(t); \quad u(1, t) = q(t); \quad t > 0; \quad (5.3.1d)$$

and

$$v_t(x,t) - v_{xx}(x,t) = (1-x)g_1(u(x,t)) + xg_2(v(x,t)) + \gamma(x,t); \quad (5.3.2a)$$

$$x \in (0,1), \quad t > 0;$$

$$v_x(0,t) = 0, \quad v_x(1,t) = 0; \quad t > 0; \quad (5.3.2b)$$

$$v(x,0) = u_0(x); \quad x \in [0,1]; \quad (5.3.2c)$$

$$v(0,t) = \theta(t); \quad v(1,t) = q(t); \quad t > 0; \quad (5.3.2d)$$

The difference between the solutions of the nonhomogeneous equations (5.3.1) and (5.3.2) is given by

$$u - v = \int_0^t \int_0^1 K(x,y,t-\tau) \left[ (1-y)(f_1(u) - g_1(v)) + y(f_2(u) - g_2(v)) \right] dy d\tau \quad (5.3.3)$$

Where  $K(x,y,t)$  is the Green's function for the Neumann problem. Now differentiating equation (5.3.3) twice with respect to  $x$  one obtains,

$$u_{xx} - v_{xx} = \int_0^t \int_0^1 K_{xx}(x,y,t-\tau) \left[ (1-y) \left( f_1(u) - g_1(v) \right) + y \left( f_2(u) - g_2(v) \right) \right] dy d\tau \quad (5.3.4)$$

According to the formula (4.3.6) in chapter IV, equations (5.1.1) and (5.1.2) can be written as

$$\begin{aligned} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= T_{\theta,q} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \theta'(t) - u_{xx}(0,t;f_1,f_2) - \gamma(0,t), \\ q'(t) - u_{xx}(1,t;f_1,f_2) - \gamma(1,t) \end{bmatrix} \end{aligned} \quad (5.3.5)$$

and

$$\begin{aligned} \begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix} &= T_{\theta,q} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \theta'(t) - v_{xx}(0,t;g_1,g_2) - \gamma(0,t), \\ q'(t) - v_{xx}(1,t;g_1,g_2) - \gamma(1,t) \end{bmatrix} \end{aligned} \quad (5.3.6)$$

Therefore,

$$\begin{aligned}
T_{\theta,q} \begin{bmatrix} f_1, f_2 \end{bmatrix} - T_{\theta,q} \begin{bmatrix} g_1, g_2 \end{bmatrix} &= \begin{bmatrix} f_1 - \tilde{g}_1, f_2 - \tilde{g}_2 \end{bmatrix} \\
&= \begin{bmatrix} v_{xx}(0,t;g_1,g_2) - u_{xx}(0,t;f_1,f_2), \\ v_{xx}(1,t;g_1,g_2) - u_{xx}(1,t;f_1,f_2) \end{bmatrix} \quad (5.3.7)
\end{aligned}$$

Substituting (5.3.4) into (5.3.7), we get

$$\begin{aligned}
&\begin{bmatrix} f_1 - \tilde{g}_1, f_2 - \tilde{g}_2 \end{bmatrix} = \\
&\left[ \int_0^t \int_0^1 K_{xx}(0,y,t-\tau) \left[ (1-y) \left( g_1(v(y,\tau)) - f_1(u(y,\tau)) \right) \right. \right. \\
&\quad \left. \left. + y \left( g_2(v(y,\tau)) - f_2(u(y,\tau)) \right) \right] dy d\tau, \right. \\
&\left. \int_0^t \int_0^1 K_{xx}(1,y,t-\tau) \left[ (1-y) \left( g_1(v(y,\tau)) - f_1(u(y,\tau)) \right) \right. \right. \\
&\quad \left. \left. + y \left( g_2(v(y,\tau)) - f_2(u(y,\tau)) \right) \right] dy d\tau \right] \quad (5.3.8)
\end{aligned}$$

We first of all obtain an estimate of the sup-norm of  $f_1 - \tilde{g}_1$  and note that  $f_2 - \tilde{g}_2$  can be estimated analogously.

$$\begin{aligned}
f_1 - \tilde{g}_1 &= \int_0^t \int_0^1 K_{xx}(0,y,t-\tau) \left( g_1(v(y,\tau)) - f_1(u(y,\tau)) \right) dy d\tau \\
&\quad + \int_0^t \int_0^1 K_{xx}(0,y,t-\tau) y \left\{ \left( g_2(v(y,\tau)) - f_2(u(y,\tau)) \right) \right. \\
&\quad \left. - \left( g_1(v(y,\tau)) - f_1(u(y,\tau)) \right) \right\} dy d\tau \\
&= I_1 + I_2 \quad (5.3.9)
\end{aligned}$$

Adding and subtracting some suitable terms,  $I_1$  as given below can be rewritten as a sum of  $I_{1,1}$  and  $I_{1,2}$ .

$$\begin{aligned}
I_1 &= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
&= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ g_1(v(y, \tau)) - f_1(v(y, \tau)) \right] dy d\tau \\
&\quad + \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ f_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
&= I_{1,1} + I_{1,2}
\end{aligned} \tag{5.3.10}$$

Consider  $f_1$  and  $g_1 \in C^\alpha$ . Making use of the identity property,  $I_{1,1}$  is estimated in the following manner.

$$\begin{aligned}
I_{1,1} &= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ g_1(v(y, \tau)) - f_1(v(y, \tau)) - \right. \\
&\quad \left. g_1(v(0, \tau)) + f_1(v(0, \tau)) \right] dy d\tau \\
|I_{1,1}| &\leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |f_1 - g_1|_\alpha |v(y, \tau) - v(0, \tau)|^\alpha dy d\tau \\
&\leq |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| (y^2/2)^\alpha dy d\tau \\
&\leq |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha 2^{-(\alpha+1)} \frac{1}{\alpha} C_3(0, 0, 2\alpha) t^\alpha
\end{aligned} \tag{5.3.11}$$

Where  $C_3(\dots)$  is defined in Appendix B (B6). Noting that  $f_1$  satisfies the property S with constant  $C_s$  and that  $I_{1,2}$  is equivalent to

$$\begin{aligned}
I_{1,2} &= - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ f_1(u(y, \tau)) - f_1(v(y, \tau)) \right] dy d\tau \\
&\quad - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ f_1(v(0, \tau)) - f_1(u(0, \tau)) \right] dy d\tau
\end{aligned}$$

we have

$$\begin{aligned}
 |I_{1,2}| &\leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| \left( y |f_1(u(\cdot, \tau)) - f_1(v(\cdot, \tau))|_1 \right) dy d\tau \\
 &\leq C_s \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| \left( y |u(\cdot, \tau) - v(\cdot, \tau)|_1 \right) dy d\tau \\
 &\leq C_s \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| \left( y |u_x(\cdot, \tau) - v_x(\cdot, \tau)|_\infty \right) dy d\tau \\
 &\leq C_s |u_x - v_x|_\infty C_3(0, 0, 1) |t|^{(1/2)}
 \end{aligned} \tag{5.3.12}$$

To compute the first part of (5.3.8), we need to estimate  $I_2$ , where

$$\begin{aligned}
 I_2 &= \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left( g_2(v(y, \tau)) - f_2(u(y, \tau)) \right) dy d\tau \\
 &\quad + \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left( f_1(u(y, \tau)) - g_1(v(y, \tau)) \right) dy d\tau \\
 &= I_{2,1} + I_{2,2}
 \end{aligned} \tag{5.3.13}$$

Once we have an estimate of  $I_{2,1}$ , the estimate for  $I_{2,2}$  will follow similarly. Adding and subtracting  $y f_2(v(y, \tau))$  inside the integral sign,  $I_{2,1}$  written as sum of two terms.

$$\begin{aligned}
 I_{2,1} &= - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left( f_2(u(y, \tau)) - f_2(v(y, \tau)) \right) dy d\tau \\
 &\quad - \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) y \left( f_2(v(y, \tau)) - g_2(v(y, \tau)) \right) dy d\tau \\
 &= I_{2,1,1} + I_{2,1,2}
 \end{aligned} \tag{5.3.14}$$

Taking modulus on both the sides of  $I_{2,1,2}$  we have,

$$\begin{aligned}
 |I_{2,1,2}| &\leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |y| |f_2(v(y, \tau)) - g_2(v(y, \tau))| dy d\tau \\
 &\leq |f_2 - g_2|_{\infty} \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |y| dy d\tau \\
 &\leq |f_2 - g_2|_{\infty} C_3(0, 0, 1) t^{(1/2)}
 \end{aligned} \tag{5.3.15}$$

Let  $f_2 \in \text{Lip class}$  with constant  $C$ , then

$$\begin{aligned}
 |I_{2,1,1}| &\leq \int_0^t \int_0^1 |K_{xx}(0, y, t-\tau)| |y| |f_2(u(y, \tau)) - f_2(v(y, \tau))| dy d\tau \\
 &\leq C C_3(0, 0, 1) |u - v|_{\infty} t^{(1/2)}
 \end{aligned} \tag{5.3.16}$$

Estimating in a similar manner and taking into account that  $f_1$  satisfies both property S and is in Lip class with constant  $D$ , we have the following estimate.

$$|I_{2,2}| \leq |f_1 - g_1|_{\infty} C_3(0, 0, 1) t^{(1/2)} + D C_3(0, 0, 1) |u - v|_{\infty} t^{(1/2)} \tag{5.3.17}$$

Now putting together all the estimates (5.3.11), (5.3.12), (5.3.15), (5.3.16) and (5.3.17), the corresponding estimate for  $f_1 - \tilde{g}_1$  is given by

$$\begin{aligned}
 |f_1 - \tilde{g}_1|_{\infty} &\leq |f_1 - g_1|_{\infty} (|v_{xx}|_{\infty})^{\alpha} 2^{-(\alpha+1)} \frac{1}{\alpha} C_3(0, 0, 2\alpha) t^{\alpha} \\
 &\quad + C_s |u_x - v_x|_{\infty} C_3(0, 0, 1) t^{(1/2)} \\
 &\quad + |f_2 - g_2|_{\infty} C_3(0, 0, 1) t^{(1/2)} \\
 &\quad + C C_3(0, 0, 1) |u - v|_{\infty} t^{(1/2)}
 \end{aligned}$$

$$\begin{aligned}
& + |f_1 - g_1|_{\infty} C_3(0,0,1) t^{(1/2)} \\
& + D C_3(0,0,1) |u - v|_{\infty} t^{(1/2)}
\end{aligned} \tag{5.3.18}$$

Analogously,

$$\begin{aligned}
|f_2 - \tilde{g}_2|_{\infty} & \leq |f_1 - g_1|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} 2^{-(\alpha+1)} \frac{1}{\alpha} C_3(1,0,2\alpha) t^{\alpha} \\
& + C_s |u_x - v_x|_{\infty} C_3(1,0,1) t^{(1/2)} \\
& + |f_2 - g_2|_{\infty} C_3(1,0,1) t^{(1/2)} \\
& + C C_3(1,0,1) |u - v|_{\infty} t^{(1/2)} \\
& + |f_1 - g_1|_{\infty} C_3(1,0,1) t^{(1/2)} \\
& + D C_3(1,0,1) |u - v|_{\infty} t^{(1/2)}
\end{aligned} \tag{5.3.19}$$

To find the sup norm of  $T_{\theta,q}$ , one can add up (5.3.18) and (5.3.19). Hence

$$\begin{aligned}
|T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)|_{\infty} & = |f_1 - \tilde{g}_1|_{\infty} + |f_2 - \tilde{g}_2|_{\infty} \\
& \leq \left\{ C_3(0,0,2\alpha) + C_3(1,0,2\alpha) \right\} |f_1 - g_1|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \frac{1}{\alpha 2^{\alpha+1}} t^{(\alpha/2)} \\
& + \left\{ C_3(0,0,1) + C_3(1,0,1) \right\} C_s |u_x - v_x|_{\infty} t^{(1/2)} \\
& + \left\{ C_3(0,0,1) + C_3(1,0,1) \right\} |f_2 - g_2|_{\infty} t^{(1/2)} \\
& + \left\{ C_3(0,0,1) + C_3(1,0,1) \right\} C |u - v|_{\infty} t^{(1/2)} \\
& + \left\{ C_3(0,0,1) + C_3(1,0,1) \right\} |f_1 - g_1|_{\infty} t^{(1/2)} \\
& + \left\{ C_3(0,0,1) + C_3(1,0,1) \right\} D |u - v|_{\infty} t^{(1/2)}
\end{aligned} \tag{5.3.20}$$

In order to get the required result, one has to show,  $|u_x - v_x|_\infty$ ,  $(|v_{xx}|_\infty)^\alpha$ , and  $|u - v|_\infty$  are small quantities for small values of "t". Differentiating the equation (5.3.3) with respect to x, one finds

$$\begin{aligned}
 u_x(x,t) - v_x(x,t) &= \int_0^t \int_0^1 K_x(x,y,t-\tau) \left( f_1(u(y,\tau)) - g_1(v(y,\tau)) \right) dy d\tau \\
 &\quad + \int_0^t \int_0^1 K_x(x,y,t-\tau) y \left( f_2(u(y,\tau)) - g_2(v(y,\tau)) - \right. \\
 &\quad \left. f_1(u(y,\tau)) + g_1(v(y,\tau)) \right) dy d\tau \\
 &= I_3 + I_4
 \end{aligned} \tag{5.3.21}$$

$I_3$  can be expanded as

$$\begin{aligned}
 I_3 &= \int_0^t \int_0^1 K_x(x,y,t-\tau) \left( f_1(u(y,\tau)) - f_1(v(y,\tau)) \right) dy d\tau \\
 &\quad + \int_0^t \int_0^1 K_x(x,y,t-\tau) \left( f_1(v(y,\tau)) - g_1(v(y,\tau)) \right) dy d\tau \\
 &= I_{3,1} + I_{3,2}
 \end{aligned} \tag{5.3.22}$$

For  $f_1 \in \text{Lip class}$  with constant D,  $I_{3,1}$  is estimated as

$$\begin{aligned}
 |I_{3,1}| &\leq D |u - v|_\infty \int_0^t \int_0^1 |K_x(x,y,t-\tau)| dy d\tau \\
 &\leq D |u - v|_\infty C_2(x,0,0) t^{1/2}
 \end{aligned} \tag{5.3.23}$$

and  $I_{3,2}$  can be estimated directly as

$$\begin{aligned}
 |I_{3,2}| &\leq |f_1 - g_1|_\infty \int_0^t \int_0^1 |K_x(x,y,t-\tau)| dy d\tau \\
 &\leq |f_1 - g_1|_\infty C_2(x,0,0) t^{1/2}
 \end{aligned} \tag{5.3.24}$$

Hence combining (5.3.23) and (5.3.24), the bound for  $I_3$  is given by

$$|I_3| \leq \left( D \|u - v\|_\infty + \|f_1 - g_1\|_\infty \right) C_2(x, 0, 0) t^{1/2} \quad (5.3.25)$$

We break  $I_4$  into two parts as follows

$$\begin{aligned} I_4 &= \int_0^t \int_0^1 K_x(x, y, t-\tau) y \left( f_2(u(y, \tau)) - g_2(v(y, \tau)) \right) dy d\tau \\ &\quad + \int_0^t \int_0^1 K_x(x, y, t-\tau) y \left( g_1(v(y, \tau)) - f_1(u(y, \tau)) \right) dy d\tau \\ &= I_{4,1} + I_{4,2} \end{aligned} \quad (5.3.26)$$

We estimate  $I_{4,1}$  and  $I_{4,2}$  in the similar as way it is done for  $I_3$ . Hence,

$$\begin{aligned} I_{4,1} &= \int_0^t \int_0^1 K_x(x, y, t-\tau) y \left( f_2(u(y, \tau)) - f_2(v(y, \tau)) \right) dy d\tau \\ &\quad + \int_0^t \int_0^1 K_x(x, y, t-\tau) y \left( f_2(v(y, \tau)) - g_2(v(y, \tau)) \right) dy d\tau \\ &= I_{4,1,1} + I_{4,1,2} \end{aligned} \quad (5.3.27)$$

For  $f_2 \in \text{Lip class}$  with constant  $C$ , the following estimate for  $I_{4,1,1}$  is given

$$\begin{aligned} |I_{4,1,1}| &\leq C \|u - v\|_\infty \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |y| dy d\tau \\ &\leq C \|u - v\|_\infty \left( \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |y - x| dy d\tau + \right. \\ &\quad \left. \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |x| dy d\tau \right) \\ &\leq C \|u - v\|_\infty \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) t^{1/2} \end{aligned} \quad (5.3.28)$$

and similarly the estimate for  $I_{4,1,2}$  is given by

$$\begin{aligned}
 |I_{4,1,2}| &\leq |f_2 - g_2|_\infty \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |y| dy d\tau \\
 &\leq |f_2 - g_2|_\infty \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) t^{1/2}
 \end{aligned} \tag{5.3.29}$$

So in view of (5.3.28) and (5.3.29),  $I_{4,1}$  is estimated as

$$|I_{4,1}| \leq \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) \left( C |u - v|_\infty + |f_2 - g_2|_\infty \right) t^{1/2} \tag{5.3.30}$$

Similarly, the estimate for  $I_{4,2}$  is given by

$$|I_{4,2}| \leq \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) \left( D |u - v|_\infty + |f_1 - g_1|_\infty \right) t^{1/2} \tag{5.3.31}$$

Adding (5.3.30) and (5.3.31), the estimated value for  $I_4$  is computed as

$$\begin{aligned}
 |I_4| &\leq \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) \\
 &\quad \left( (C + D) |u - v|_\infty + |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t^{1/2}
 \end{aligned} \tag{5.3.32}$$

Hence combining (5.3.25) and (5.3.32) one obtains the required estimate for  $|u_x - v_x|_\infty$ .

$$\begin{aligned}
 &|u_x - v_x|_\infty \\
 &\leq \left[ \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) (C + D) + D C_2(x, 0, 0) \right] |u - v|_\infty t^{1/2} \\
 &+ \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| + C_2(x, 0, 0) \right) |f_1 - g_1|_\infty t^{1/2} \\
 &+ \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) |f_2 - g_2|_\infty t^{1/2}
 \end{aligned}$$

$$\begin{aligned}
&= \Lambda_1 |u - v|_\infty t^{1/2} + \Lambda_2 |f_1 - g_1|_\infty t^{1/2} + \Lambda_3 |f_2 - g_2|_\infty t^{1/2} \\
&\leq \Lambda_1 |u - v|_\infty t^{1/2} + \Lambda_2 \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t^{1/2}
\end{aligned} \tag{5.3.33}$$

Now we obtain an estimate for  $|u - v|_\infty$ . Indeed

$$\begin{aligned}
u(x, t) - v(x, t) &= \int_0^t \int_0^1 K(x, y, t-\tau) \left( f_1(u(y, \tau)) - g_1(v(y, \tau)) \right) dy d\tau \\
&\quad + \int_0^t \int_0^1 K(x, y, t-\tau) y \left( f_2(u(y, \tau)) - g_2(v(y, \tau)) - \right. \\
&\quad \left. f_1(u(y, \tau)) + g_1(v(y, \tau)) \right) dy d\tau \\
&= I_5 + I_6
\end{aligned} \tag{5.3.34}$$

Since the very structure of  $u - v$  is similar to  $u_x - v_x$ , instead of the kernel which is of first derivative of  $x$ , the estimate for (5.3.34) given similar to (5.3.33) except some changes in the domain constants using the Appendix B (B1).

$$\begin{aligned}
\int_0^t \int_0^1 |K(x, y, t-\tau)| dy d\tau &\leq C_1(x, 0, 0) t, \text{ and} \\
\int_0^t \int_0^1 |K(x, y, t-\tau)| |y - x| dy d\tau &\leq C_1(x, 0, 1) t^{3/2}
\end{aligned}$$

Hence

$$\begin{aligned}
|u - v|_\infty &\leq \Lambda_4 |u - v|_\infty t + \Lambda_5 |f_1 - g_1|_\infty t + \Lambda_6 |f_2 - g_2|_\infty t \\
&\leq \Lambda_4 |u - v|_\infty t + \Lambda_5 \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t \\
\Rightarrow |u - v|_\infty &\leq \frac{\Lambda_5}{1 - \Lambda_4 t} \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t \\
&= \Lambda_7 \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t
\end{aligned} \tag{5.3.35}$$

Substituting (5.3.35) in (5.3.33), one obtains

$$|u_x - v_x|_\infty \leq \left( \Lambda_1 \Lambda_7 t + \Lambda_2 \right) \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t^{1/2} \quad (5.3.36)$$

Finally we estimate  $(|v_{xx}|_\infty)^\alpha$ . Utilizing the representation, one can write

$$\begin{aligned} v_{xx} = \psi_{xx} + \int_0^t \int_0^1 K_{xx}(x, y, t-\tau) g_1(v(y, \tau)) dy d\tau \\ + \int_0^t \int_0^1 K_{xx}(x, y, t-\tau) y \left[ g_2(v(y, \tau)) - g_1(v(y, \tau)) \right] dy d\tau \end{aligned} \quad (5.3.37)$$

Where  $\psi$  is the comparison function. Similar procedure can be adopted to bound (5.3.37) as it is done for  $(|u_{xx}|_\infty)^\alpha$  in chapter IV to obtain analogous estimate.

#### 5.4 SEMINORM ESTIMATION :

In order to complete the  $\alpha$ -norm, in this section we obtain an estimate of the  $\alpha$ -seminorm of the map  $T_{\theta, q}$ . According to the definition it can be written as

$$\begin{aligned} |T_{\theta, q}(f_1, f_2) - T_{\theta, q}(g_1, g_2)|_\alpha &= |(f_1, f_2) - (\tilde{g}_1, \tilde{g}_2)|_\alpha \\ &= \left( |f_1 - \tilde{g}_1|_\alpha + |f_2 - \tilde{g}_2|_\infty \right) \end{aligned} \quad (5.4.1)$$

We estimate  $|f_1 - \tilde{g}_1|_\alpha$ , the estimate for  $|f_2 - \tilde{g}_2|_\alpha$  will follow similarly. Without loss of generality assume  $t_1 > t_2$ . Then

$$\begin{aligned} |f_1 - \tilde{g}_1|_\alpha = \\ \sup_{\theta(t_1) \neq \theta(t_2)} \frac{\left| \left[ f_1(\theta(t_1)) - \tilde{g}_1(\theta(t_1)) \right] - \left[ f_1(\theta(t_2)) - \tilde{g}_1(\theta(t_2)) \right] \right|}{|\theta(t_1) - \theta(t_2)|^\alpha} \end{aligned} \quad (5.4.2)$$

It is given that

$$\begin{aligned}
 & \left[ f_1 - \tilde{g}_1, f_2 - \tilde{g}_2 \right] = \\
 & \left[ \int_0^t \int_0^1 K_{xx}(0, y, t-\tau) \left[ (1-y) \left\{ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right\} \right. \right. \\
 & \quad \left. \left. + y \left\{ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right\} \right] dy d\tau, \right. \\
 & \left. \int_0^t \int_0^1 K_{xx}(1, y, t-\tau) \left[ (1-y) \left\{ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right\} \right. \right. \\
 & \quad \left. \left. + y \left\{ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right\} dy d\tau \right] \right] \quad (5.4.3)
 \end{aligned}$$

In order to get the  $|f_1 - \tilde{g}_1|_\alpha$ , our aim is to compute the numerator of (5.4.2). The limits of the integrations are divided into two parts, from 0 to  $t_2$  and from  $t_2$  to  $t_1$ . Thus,

$$\begin{aligned}
 I_7 &= \left[ f_1(\theta(t_1)) - \tilde{g}_1(\theta(t_1)) \right] - \left[ f_1(\theta(t_2)) - \tilde{g}_1(\theta(t_2)) \right] \\
 &= \int_0^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
 &+ \int_0^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ \left\{ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right\} - \right. \\
 &\quad \left. \left\{ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right\} \right] dy d\tau \\
 &- \int_0^{t_2} \int_0^1 K_{xx}(0, y, t_2 - \tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
 &- \int_0^{t_2} \int_0^1 K_{xx}(0, y, t_2 - \tau) y \left[ \left\{ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right\} - \right. \\
 &\quad \left. \left\{ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right\} \right] dy d\tau
 \end{aligned}$$

Now the R. H. S. of the above expression can be broken into two parts with respect to the limits of integrations. Indeed,

$$\begin{aligned}
 I_7 = & \left[ \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \right. \\
 & + \int_0^{t_2} \int_0^1 \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] \times \\
 & \quad \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
 & + \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ g_2(v(y, \tau)) - f_2(u(y, \tau)) - \right. \\
 & \quad \left. g_1(v(y, \tau)) + f_1(u(y, \tau)) \right] dy d\tau \\
 & + \int_0^{t_2} \int_0^1 \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] y \times \\
 & \quad \left[ g_2(v(y, \tau)) - f_2(u(y, \tau)) - g_1(v(y, \tau)) + f_1(u(y, \tau)) \right] dy d\tau \left. \right] \\
 & \hspace{15em} (5.4.4)
 \end{aligned}$$

These four parts can be represented as  $I_8$ ,  $I_9$ ,  $I_{10}$  and  $I_{11}$  respectively. Now we proceed to estimate each of these terms :

$$\begin{aligned}
 I_8 = & \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
 = & \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ g_1(v(y, \tau)) - f_1(v(y, \tau)) \right] dy d\tau \\
 & + \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ f_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
 = & I_{8,1} + I_{8,2} \\
 & \hspace{15em} (5.4.5)
 \end{aligned}$$

Let  $f_1, g_1 \in C^\alpha$ . By the application of the identity property  $I_{8,1}$  can be rewritten as :

$$I_{8,1} = \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ \left\{ g_1(v(y, \tau)) - f_1(v(y, \tau)) \right\} - \left\{ g_1(v(0, \tau)) - f_1(v(0, \tau)) \right\} \right] dy d\tau$$

This will imply,

$$|I_{8,1}| \leq |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \frac{1}{\alpha} 2^{-(\alpha+1)} C_3(0,0,2\alpha) |t_1 - t_2|^\alpha \quad (5.4.6)$$

To estimate  $I_{8,2}$  use the fact that  $f_1$  satisfies the property S. Making use of the identity property,  $I_{8,2}$  is first of all written as

$$I_{8,2} = \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) \left[ f_1(v(y, \tau)) - f_1(u(y, \tau)) - f_1(v(0, \tau)) + f_1(u(0, \tau)) \right] dy d\tau$$

Following the arguments utilized for the estimation of  $I_{1,2}$  and Appendix B (B5), we estimate  $I_{8,2}$  as

$$\begin{aligned} |I_{8,2}| &\leq \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| \left\{ y |f_1(u(\cdot, \tau)) - f_1(v(\cdot, \tau))| \right\} dy d\tau \\ &\leq C_s \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |u(\cdot, \tau) - v(\cdot, \tau)|_1 |y| dy d\tau \\ &\leq C_s \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |u_x(\cdot, \tau) - v_x(\cdot, \tau)|_\infty |y| dy d\tau \\ &\leq C_s |u_x - v_x|_\infty C_3(0,0,1) |t_1 - t_2|^{1/2} \end{aligned} \quad (5.4.7)$$

From (5.4.6) and (5.4.7), the estimate for  $I_8$  is

$$\begin{aligned}
|I_8| &\leq |f_1 - g_1|_\infty (|v_{xx}|_\infty)^\alpha \frac{1}{\alpha} 2^{-(\alpha+1)} C_3(0,0,2\alpha) |t_1 - t_2|^\alpha \\
&+ C_\# |u_x - v_x|_\infty C_3(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.8}$$

$I_{10}$  can be estimated in a similar fashion, as it is done for  $I_8$ . Indeed,

$$\begin{aligned}
I_{10} &= \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right] dy d\tau \\
&- \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
&= I_{10,1} + I_{10,2}
\end{aligned} \tag{5.4.9}$$

Since we can not apply the identity property at  $x = 0$ ,  $I_{10,1}$  and  $I_{10,2}$  are estimated directly.

$$\begin{aligned}
I_{10,1} &= - \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ f_2(v(y, \tau)) - g_2(v(y, \tau)) \right] dy d\tau \\
&- \int_{t_2}^{t_1} \int_0^1 K_{xx}(0, y, t_1 - \tau) y \left[ f_2(u(y, \tau)) - f_2(v(y, \tau)) \right] dy d\tau \\
&= I_{10,1,1} + I_{10,1,2}
\end{aligned} \tag{5.4.10}$$

$I_{10,1,1}$  is estimated as,

$$\begin{aligned}
|I_{10,1,1}| &\leq |f_2 - g_2|_\infty \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |y| dy d\tau \\
&\leq |f_2 - g_2|_\infty C_3(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.11}$$

Let  $f_2 \in \text{Lip}$  class with constant  $C$ . Then the bound for  $I_{10,1,2}$  is given by

$$|I_{10,1,2}| \leq \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |y| |f_2(u(y, \tau)) - f_2(v(y, \tau))| dy d\tau$$

$$\begin{aligned}
&\leq C \int_{t_2}^{t_1} \int_0^1 |K_{xx}(0, y, t_1 - \tau)| |y| |u(y, \tau) - v(y, \tau)| dy d\tau \\
&\leq C C_3(0, 0, 1) \|u - v\|_{\infty} |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.12}$$

From (5.4.11) and (5.4.12), the estimated value for  $I_{10,1}$  is given by

$$\begin{aligned}
|I_{10,1}| &\leq \|f_2 - g_2\|_{\infty} C_3(0, 0, 1) |t_1 - t_2|^{1/2} \\
&+ C C_3(0, 0, 1) \|u - v\|_{\infty} |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.13}$$

For  $f_1 \in \text{Lip}$  class with Lipschitz constant  $D$ ,  $I_{10,2}$  is estimated in the same way as it is done for  $I_{10,1}$ . So the result can be written as

$$\begin{aligned}
|I_{10,2}| &\leq \|f_1 - g_1\|_{\infty} C_3(0, 0, 1) |t_1 - t_2|^{1/2} \\
&+ D C_3(0, 0, 1) \|u - v\|_{\infty} |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.14}$$

Hence from the results that are obtained in (5.4.13) and (5.4.14), the estimate for  $I_{10}$  is given by

$$\begin{aligned}
|I_{10}| &\leq |I_{10,1}| + |I_{10,2}| \\
&\leq \|f_2 - g_2\|_{\infty} C_3(0, 0, 1) |t_1 - t_2|^{1/2} \\
&+ C C_3(0, 0, 1) \|u - v\|_{\infty} |t_1 - t_2|^{1/2} \\
&+ \|f_1 - g_1\|_{\infty} C_3(0, 0, 1) |t_1 - t_2|^{1/2} \\
&+ D C_3(0, 0, 1) \|u - v\|_{\infty} |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.15}$$

The expressions for  $I_9$  and  $I_{11}$ , which are almost similar can be estimated by the application of Mean value theorem.  $I_9$  is written as

$$\begin{aligned}
I_9 &= \int_0^{t_2} \int_0^1 \left[ K_{xx}(0, y, t_1 - \tau) - K_{xx}(0, y, t_2 - \tau) \right] \cdot \\
&\quad \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau \\
&= \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 K_{xxt}(0, y, s - \tau) \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau ds \\
&= \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 K_{xxt}(0, y, s - \tau) \left[ g_1(v(y, \tau)) - f_1(v(y, \tau)) \right] dy d\tau ds \\
&\quad + \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 K_{xxt}(0, y, s - \tau) \left[ f_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau ds \\
&= I_{9,1} + I_{9,2} \tag{5.4.16}
\end{aligned}$$

Let  $f_1, g_1 \in C^\alpha$ , then application of the identity property will give us the following equivalent expression for  $I_{9,1}$ .

$$\begin{aligned}
I_{9,1} &= \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 K_{xxt}(0, y, s - \tau) \left[ g_1(v(y, \tau)) - f_1(v(y, \tau)) \right. \\
&\quad \left. - g_1(v(0, \tau)) + f_1(v(0, \tau)) \right] dy d\tau ds
\end{aligned}$$

Taking modulus on both the sides and estimating, we have

$$\begin{aligned}
|I_{9,1}| &\leq \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s - \tau)| |f_1 - g_1|_\alpha |v(y, \tau) - v(0, \tau)|^\alpha dy d\tau ds \\
&\leq 2^{-\alpha} |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s - \tau)| |y|^{2\alpha} dy d\tau ds \\
&\leq 2^{-\alpha} |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha C_4(0, 0, 2\alpha) \int_{t_2}^{t_1} \int_0^{t_2} |s - \tau|^{\alpha-2} d\tau ds \tag{from (B7)}
\end{aligned}$$

$$\begin{aligned}
|I_9| &\leq |I_{9,1}| + |I_{9,2}| \\
&\leq 2^{-\alpha} |f_1 - g_1|_{\alpha} (|v_{xx}|_{\infty})^{\alpha} \frac{C_4(0,0,2\alpha)}{\alpha(1-\alpha)} |t_1 - t_2|^{\alpha} \\
&\quad + 4C_{\alpha} |u_x - v_x|_{\infty} C_4(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.19}$$

Similarly, using the Mean value theorem, the equivalent form for  $I_{11}$  is expressed as

$$\begin{aligned}
I_{11} &= \int_{t_2}^1 \int_0^2 \int_0^1 K_{xxt}(0, y, s-\tau) y \left[ g_2(v(y, \tau)) - f_2(u(y, \tau)) \right] dy d\tau ds \\
&\quad - \int_{t_2}^1 \int_0^2 \int_0^1 K_{xxt}(0, y, s-\tau) y \left[ g_1(v(y, \tau)) - f_1(u(y, \tau)) \right] dy d\tau ds \\
&= I_{11,1} + I_{11,2}
\end{aligned} \tag{5.4.20}$$

$I_{11,1}$  is estimated first. The result obtained is on same line as  $I_{11,2}$ . With suitable choice of terms,  $I_{11,1}$  is written in an equivalent form.

$$\begin{aligned}
I_{11,1} &= - \int_{t_2}^1 \int_0^2 \int_0^1 K_{xxt}(0, y, s-\tau) y \left[ f_2(v(y, \tau)) - g_2(v(y, \tau)) \right] dy d\tau ds \\
&\quad - \int_{t_2}^1 \int_0^2 \int_0^1 K_{xxt}(0, y, s-\tau) y \left[ f_2(u(y, \tau)) - f_2(v(y, \tau)) \right] dy d\tau ds \\
&= I_{11,1,1} + I_{11,1,2}
\end{aligned} \tag{5.4.21}$$

$I_{11,1,1}$  is estimated as

$$|I_{11,1,1}| \leq \int_{t_2}^1 \int_0^2 \int_0^1 |K_{xxt}(0, y, s-\tau)| |y| |f_2(v(y, \tau)) - g_2(v(y, \tau))| dy d\tau ds$$

$$\begin{aligned}
&\leq |f_2 - g_2|_\infty \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s-\tau)| |y| dy d\tau ds \\
&\leq 4 |f_2 - g_2|_\infty C_4(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.22}$$

If  $f_2 \in \text{Lip class}$ , then the bound for  $I_{11,1,2}$  is given by

$$\begin{aligned}
|I_{11,1,2}| &\leq \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s-\tau)| |y| |f_2(u(y, \tau)) - f_2(v(y, \tau))| dy d\tau ds \\
&\leq C \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s-\tau)| |u(y, \tau) - v(y, \tau)| |y| dy d\tau ds \\
&\leq C |u - v|_\infty \int_{t_2}^{t_1} \int_0^{t_2} \int_0^1 |K_{xxt}(0, y, s-\tau)| |y| dy d\tau ds \\
&\leq 4C |u - v|_\infty C_4(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.23}$$

Now we sum up inequalities (5.4.22) and (5.4.23) to obtain the following bound for  $I_{11,1}$ .

$$\begin{aligned}
|I_{11,1}| &\leq |I_{11,1,1}| + |I_{11,1,2}| \\
&\leq 4 \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] C_4(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.24}$$

Now taking  $f_1 \in \text{Lip class}$  with Lipschitz constant  $D$ , the estimate for  $I_{11,2}$  is given by

$$|I_{11,2}| \leq 4 \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] C_4(0,0,1) |t_1 - t_2|^{1/2} \tag{5.4.25}$$

Thus from (5.4.24) and (5.4.25), one obtains

$$\begin{aligned}
|I_{11}| &\leq |I_{11,1}| + |I_{11,2}| \\
&\leq 4 \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] C_4(0,0,1) |t_1 - t_2|^{1/2} \\
&\quad + 4 \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] C_4(0,0,1) |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.26}$$

Hence from (5.4.8), (5.4.15), (5.4.19) and (5.4.26) one obtains the estimate for  $I_7$  as given below.

$$\begin{aligned}
|I_7| &= \left| \left[ f_1(\theta(t_1)) - \tilde{g}_1(\theta(t_1)) \right] - \left[ f_1(\theta(t_2)) - \tilde{g}_1(\theta(t_2)) \right] \right| \\
&\leq |I_8| + |I_9| + |I_{10}| + |I_{11}| \\
&\leq |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \frac{1}{\alpha} 2^{-(\alpha+1)} C_3(0,0,2\alpha) |t_1 - t_2|^\alpha \\
&\quad + C_s |u_x - v_x|_\infty C_3(0,0,1) |t_1 - t_2|^{1/2} \\
&\quad + 2^{-\alpha} |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \frac{C_4(0,0,2\alpha)}{\alpha(1-\alpha)} |t_1 - t_2|^{1/2} \\
&\quad + 4C_s |u_x - v_x|_\infty C_4(0,0,1) |t_1 - t_2|^{1/2} \\
&\quad + C_3(0,0,1) \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] |t_1 - t_2|^{1/2} \\
&\quad + C_3(0,0,1) \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] |t_1 - t_2|^{1/2} \\
&\quad + 4C_4(0,0,1) \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] |t_1 - t_2|^{1/2} \\
&\quad + 4C_4(0,0,1) \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] |t_1 - t_2|^{1/2}
\end{aligned} \tag{5.4.27}$$

Denoting  $C_a(0,0,\alpha) = 2^{-(\alpha+1)} (1-\alpha) C_3(0,0,2\alpha) + 2^{-\alpha} C_4(0,0,2\alpha)$

$$C_b(0,0,1) = C_s C_3(0,0,1) + 4 C_s C_4(0,0,1) \tag{5.4.28}$$

We obtain

$$\begin{aligned}
 |I_7| \leq & \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha |t_1 - t_2|^\alpha \\
 & + C_b(0,0,1) |u_x - v_x|_\infty |t_1 - t_2|^{1/2} \\
 & + C_b(0,0,1) \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] |t_1 - t_2|^{1/2} \\
 & + C_b(0,0,1) \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] |t_1 - t_2|^{1/2}
 \end{aligned} \tag{5.4.29}$$

Now we proceed to evaluate the  $\alpha$ -semi norm of  $|f_1 - \tilde{g}_1|$ . For that purpose (5.4.29) is divided by  $|\theta(t_1) - \theta(t_2)|^\alpha$ .

$$\begin{aligned}
 \text{Therefore, } & \frac{\left| \left[ f_1(\theta(t_1)) - \tilde{g}_1(\theta(t_1)) \right] - \left[ f(\theta(t_2)) - \tilde{g}_1(\theta(t_2)) \right] \right|}{|\theta(t_1) - \theta(t_2)|^\alpha} \\
 \leq & \left[ \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1 - g_1|_\alpha (|v_{xx}|_\infty)^\alpha \right. \\
 & + C_b(0,0,1) |u_x - v_x|_\infty |t_1 - t_2|^{(1/2)-\alpha} \\
 & + C_b(0,0,1) \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] |t_1 - t_2|^{(1/2)-\alpha} \\
 & \left. + C_b(0,0,1) \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] |t_1 - t_2|^{(1/2)-\alpha} \right] \frac{|t_1 - t_2|^\alpha}{|\theta(t_1) - \theta(t_2)|^\alpha}
 \end{aligned}$$

Hence

$$\begin{aligned}
|f_1 - \tilde{g}_1|_\alpha &\leq \frac{(|v_{xx}|_\infty)^\alpha}{(\inf|\theta'|)^\alpha} \left[ \frac{C_a(0,0,\alpha)}{\alpha(1-\alpha)} |f_1 - g_1|_\alpha \right] \\
&+ \frac{1}{(\inf|\theta'|)^\alpha} \left[ |u_x - v_x|_\infty + \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] \right. \\
&\left. + \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] \right] C_b(0,0,1) |t_1 - t_2|^{(1/2)-\alpha}
\end{aligned}
\tag{5.4.30}$$

Equation (5.4.3) expresses the exact expression for  $f_2 - \tilde{g}_2$ . One can see that an estimate similar to the expression (5.4.30) will be obtained for  $|f_2 - \tilde{g}_2|_\alpha$  as well. Here the only difference is that the kernel is evaluated at  $X = 1$ . Therefore, the estimate has the form

$$\begin{aligned}
|f_2 - \tilde{g}_2|_\alpha &\leq \left( \frac{|v_{xx}|_\infty}{\inf|\theta'|} \right)^\alpha \left[ \frac{C_a(1,0,\alpha)}{\alpha(1-\alpha)} |f_1 - g_1|_\alpha \right] \\
&+ \frac{1}{(\inf|\theta'|)^\alpha} \left[ |u_x - v_x|_\infty + \left[ |f_2 - g_2|_\infty + C |u - v|_\infty \right] \right. \\
&\left. + \left[ |f_1 - g_1|_\infty + D |u - v|_\infty \right] \right] C_b(1,0,1) |t_1 - t_2|^{(1/2)-\alpha}
\end{aligned}
\tag{5.4.31}$$

Equation (5.4.1) implies the following estimate :

$$\begin{aligned}
|T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)|_\alpha &= \left( |f_1 - \tilde{g}_1|_\alpha + |f_2 - \tilde{g}_2|_\alpha \right) \\
&\leq \left( \frac{|v_{xx}|_\infty}{\inf|\theta'|} \right)^\alpha \left[ C_c(0,\alpha) |f_1 - g_1|_\alpha \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(\inf |\theta'|)^\alpha} \left[ |u_x - v_x|_\infty + \left\{ |f_2 - g_2|_\infty + C |u - v|_\infty \right\} \right. \\
& \left. + \left\{ |f_1 - g_1|_\infty + D |u - v|_\infty \right\} \right] C_c(0,1) |t_1 - t_2|^{(1/2)-\alpha} \quad (5.4.32)
\end{aligned}$$

Where

$$C_c(0, \alpha) = \frac{C_a(0, 0, \alpha)}{\alpha(1 - \alpha)} + \frac{C_a(1, 0, \alpha)}{\alpha(1 - \alpha)}$$

$$C_c(0, 1) = C_b(0, 0, 1) + C_b(1, 0, 1) \quad (5.4.33)$$

The estimates for  $|u_x - v_x|_\infty$  and  $|u - v|_\infty$  can be given by (5.3.36) and (5.3.35) respectively, i.e.,

$$|u_x - v_x|_\infty \leq \left( \Lambda_1 \Lambda_7 t_1 + \Lambda_2 \right) \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t_1^{1/2} \quad (5.4.34)$$

$$|u - v|_\infty \leq \Lambda_7 \left( |f_1 - g_1|_\infty + |f_2 - g_2|_\infty \right) t_1 \quad (5.4.35)$$

Now we are in a position to write the complete expression for the  $\alpha$ -norm of  $T_{\theta,q}$ .

$$\begin{aligned}
\|T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)\|_\alpha &= |T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)|_\infty \\
&+ |T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)|_\alpha \quad (5.4.36)
\end{aligned}$$

Substituting (5.3.35) and (5.3.36) in (5.3.20) and abbreviating the constants, the 1st term of R. H. S. of the above expression is repeated for convenience,

$$\begin{aligned}
& |T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)|_\infty \\
& \leq \Lambda_8 |f_1 - g_1|_\alpha t^{\alpha/2}
\end{aligned}$$

$$\begin{aligned}
& + \Lambda_9 \left( \Lambda_1 \Lambda_7 t_1 + \Lambda_2 \right) \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) t \\
& + \Lambda_{10} \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) t^{1/2} \\
& + \Lambda_{11} \Lambda_7 \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) t^{3/2}
\end{aligned} \tag{5.4.37}$$

Similarly, substituting (5.4.34) and (5.4.35) in (5.4.32) the second term of R. H. S. of (5.4.36) becomes

$$\begin{aligned}
& \|T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)\|_\alpha \leq \Lambda_{12} \|f_1 - g_1\|_\alpha \\
& + \Lambda_{13} \left( \Lambda_1 \Lambda_7 t_1 + \Lambda_2 \right) \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) t_1^{1/2} |t_1 - t_2|^{(1/2)-\alpha} \\
& + \Lambda_{13} \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) |t_1 - t_2|^{(1/2)-\alpha} \\
& + \Lambda_{14} \Lambda_7 \left( \|f_1 - g_1\|_\infty + \|f_2 - g_2\|_\infty \right) t_1^{1/2} |t_1 - t_2|^{(1/2)-\alpha}
\end{aligned} \tag{5.4.38}$$

Considering individual norm to be less than the  $\alpha$ -norm, (5.4.37) and (5.4.38) together will give us

$$\|T_{\theta,q}(f_1, f_2) - T_{\theta,q}(g_1, g_2)\|_\alpha \leq \Lambda_{15} \left( \|f_1 - g_1\|_\alpha + \|f_2 - g_2\|_\alpha \right) \tag{5.4.39}$$

$$\text{Where } \Lambda_{15} = \left( \Lambda_8 t^{\alpha/2} + \Lambda_{12} \right)$$

$$\begin{aligned}
& + \left( \Lambda_9 \left( \Lambda_1 \Lambda_7 t + \Lambda_2 \right) t \right. \\
& + \Lambda_{13} \left( \Lambda_1 \Lambda_7 t_1 + \Lambda_2 \right) t_1^{1/2} |t_1 - t_2|^{(1/2)-\alpha} \Big) \\
& + \left( \Lambda_{10} t^{1/2} + \Lambda_{13} |t_1 - t_2|^{(1/2)-\alpha} \right) \\
& + \left( \Lambda_{11} \Lambda_7 t^{3/2} + \Lambda_{14} \Lambda_7 t^{1/2} |t_1 - t_2|^{(1/2)-\alpha} \right)
\end{aligned} \tag{5.4.40}$$

and therefore, the map  $T_{\theta,q}$  is a contraction if  $\Lambda_{15} < 1$ .

Let us repeat  $\Lambda_i$ ,  $i = 1, \dots, 15$  for convenience.

$$\Lambda_1 = \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right) (C + D) + D C_2(x, 0, 0)$$

$$\Lambda_2 = \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| + C_2(x, 0, 0) \right)$$

$$\Lambda_3 = \left( C_2(x, 0, 1) t^{1/2} + C_2(x, 0, 0) |x| \right)$$

$$\Lambda_4 = \left( C_1(x, 0, 1) t^{1/2} + C_1(x, 0, 0) |x| \right) (C + D) + D C_1(x, 0, 0)$$

$$\Lambda_5 = \left( C_1(x, 0, 1) t^{1/2} + C_1(x, 0, 0) |x| + C_1(x, 0, 0) \right)$$

$$\Lambda_6 = \left( C_1(x, 0, 1) t^{1/2} + C_1(x, 0, 0) |x| \right)$$

$$\Lambda_7 = \frac{\Lambda_5}{1 - \Lambda_4 t} \quad \Lambda_8 = \left( C_3(0, 0, 2\alpha) + C_3(1, 0, 2\alpha) \right) (|v_{xx}|_\infty)^\alpha 2^{-(\alpha+1)} \frac{1}{\alpha}$$

$$\Lambda_9 = C_s \left( C_3(0, 0, 1) + C_3(1, 0, 1) \right) \quad \Lambda_{10} = \left( C_3(0, 0, 1) + C_3(1, 0, 1) \right)$$

$$\Lambda_{11} = 2 \left( C_3(0, 0, 1) + C_3(1, 0, 1) \right) \quad \Lambda_{12} = C_c(0, \alpha) \left( \frac{|v_{xx}|_\infty}{\inf |\theta'|} \right)^\alpha$$

$$\Lambda_{13} = C_c(0, \alpha) \frac{1}{(\inf |\theta'|)^\alpha} \quad \Lambda_{14} = (C + D) C_c(0, 1) \frac{1}{(\inf |\theta'|)^\alpha}$$

In order to obtain  $T_{\theta, q}$  to be a center contraction map we need to show that  $\left( \frac{|v_{xx}|_\infty}{\inf |\theta'|} \right)^\alpha$  is bounded. This can be achieved by utilizing the representation (5.3.37) and the steps (4.5.39)-(4.5.43) of chapter IV. Hence using the flat initial value, we can see that  $T_{\theta, q}$  is indeed a center contraction map.

Now we define the sequence of iterates  $\{p_1^{(n)}\}$  for  $i = 1, 2$  : defined by

$$p_1^{(n+1)}(\theta(t)) = T_{\theta,q} [p_1^{(n)}](t)$$

since  $f_1$  is a  $(\theta,q)$ -fixed point of  $T_{\theta,q}$ , clearly by the estimate

$$\|f_1 - p_1^{(n+1)}\|_{\alpha} \leq \Lambda_{15} \|f_1 - p_1^{(n)}\|_{\alpha}$$

and consequently  $p_1^{(n)} - f_1 \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $C^{\alpha}$  is complete,  $p_1^{(n)} \rightarrow f_1$  in  $C^{\alpha}$ . To see that  $T_{\theta,q}$  has a unique fixed point, we proceed by the method of contradiction. Suppose  $h = (h_1, h_2)$  be another fixed point of  $T_{\theta,q}$ , different from  $f$  but we know that

$$\|h - f\|_{\alpha} = \|T_{\theta}[h_1, h_2] - T_{\theta}[f_1, f_2]\|_{\alpha} \leq \Lambda_{15} \|h - f\|_{\alpha} < \|h - f\|_{\alpha}$$

This is a contradiction. This shows that  $f = h$ .

This completes the basic assertions of this chapter. ■

## 5.5 CONCLUSIONS :

In this chapter we have shown the uniqueness of a solution which is established by the method of center contraction. With the assumptions of the flat initial values the result is shown to be true for small time interval.

# APPENDIX-A

## COMPUTATION OF DOMAIN CONSTANTS

The terms of the following form are needed to be estimated.

$$I_j = \int_{t_2}^{t_1} \int_0^\infty \left| \frac{\partial^j}{\partial x^j} K(x, y, t_1 - \tau) \right| |x-y|^\alpha dy d\tau,$$

for  $j = 0, 1, 2, \dots; \quad 0 < \alpha \leq 1.$

In the case  $j = 0$ , ( $x, y$  are positive)

$$\begin{aligned} I_0 &= \int_{t_2}^{t_1} \int_0^\infty |K(x, y, t_1 - \tau)| |x-y|^\alpha dy d\tau \\ &\leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \left[ \frac{e^{-(x-y)^2/4(t_1-\tau)}}{\sqrt{t_1-\tau}} + \frac{e^{-(x+y)^2/4(t_1-\tau)}}{\sqrt{t_1-\tau}} \right] |x-y|^\alpha dy d\tau \\ &\leq \frac{1}{\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \frac{e^{-(x-y)^2/4(t_1-\tau)}}{\sqrt{t_1-\tau}} |x-y|^\alpha dy d\tau \end{aligned}$$

let  $\sigma = \frac{y-x}{2\sqrt{t_1-\tau}}$  for  $t_1 > \tau$ , then

$$I_0 \leq \frac{1}{\sqrt{\pi}} \int_{t_2}^{t_1} \int_{-\frac{x}{\sqrt{t_1-\tau}}}^\infty e^{-\sigma^2} \sigma^\alpha 2^\alpha |t_1-\tau|^{\alpha/2} d\sigma d\tau$$

$$\begin{aligned}
& \leq \frac{2^\alpha}{\sqrt{\pi}} \int_{t_2}^{t_1} |t_1 - \tau|^{\alpha/2} \int_{-\infty}^{\infty} e^{-\sigma^2} \sigma^\alpha d\sigma d\tau \\
& = \left[ \frac{2^\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\sigma^2} \sigma^\alpha d\sigma \right] \frac{(t_1 - t_2)^{(\alpha/2)+1}}{1 + (\alpha/2)} \\
& = C_1(\alpha) (t_1 - t_2)^{(\alpha/2)+1}
\end{aligned} \tag{A1}$$

$$\text{where } C_1(\alpha) = \frac{2^{\alpha+1}}{(2+\alpha)\sqrt{\pi}} \left[ \int_{-\infty}^{\infty} e^{-\sigma^2} \sigma^\alpha d\sigma \right]$$

In the case  $x = 0$ ,

$$\begin{aligned}
I_0 & \leq \frac{2^\alpha}{\sqrt{\pi}} \left[ \int_0^\infty e^{-\sigma^2} \sigma^\alpha d\sigma \right] \frac{(t_1 - t_2)^{(\alpha/2)+1}}{1 + (\alpha/2)} \\
& = (1/2) C_1(\alpha) (t_1 - t_2)^{(\alpha/2)+1}
\end{aligned} \tag{A2}$$

When  $j = 1$ ,

$$K_x(x, y, t) = \frac{-1}{2\sqrt{\pi} t} \left\{ \frac{(x-y)}{2\sqrt{t}} e^{-(x-y)^2/4t} + \frac{(x+y)}{2\sqrt{t}} e^{-(x+y)^2/4t} \right\}$$

Since  $|x - y| < |x + y|$

$$\begin{aligned}
I_1 & = \int_{t_2}^{t_1} \int_0^\infty |K_x(x, y, t_1 - \tau)| |x-y|^\alpha dy d\tau \\
& \leq \int_{t_2}^{t_1} \int_0^\infty \frac{1}{2\sqrt{\pi}} \frac{e^{-(x-y)^2/4(t_1-\tau)}}{|t_1-\tau|} \frac{|x-y|}{2\sqrt{t_1-\tau}} |x-y|^\alpha dy d\tau \\
& \quad + \int_{t_2}^{t_1} \int_0^\infty \frac{1}{2\sqrt{\pi}} \frac{e^{-(x+y)^2/4(t_1-\tau)}}{|t_1-\tau|} \frac{|x+y|}{2\sqrt{t_1-\tau}} |x+y|^\alpha dy d\tau
\end{aligned}$$

Therefore,

$$I_1 \leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4(t_1-\tau)}}{|t_1-\tau|} \frac{|x-y|^{1+\alpha}}{2\sqrt{t_1-\tau}} dy d\tau$$

with the substitution  $\sigma = \frac{y-x}{2\sqrt{t_1-\tau}}$  for  $t_1 > \tau$ ,

$$I_1 \leq \frac{2^\alpha}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_{-\infty}^{\infty} e^{-\sigma^2} \sigma^{1+\alpha} |t_1-\tau|^{(\alpha-1)/2} d\sigma d\tau$$

$$= \frac{2^\alpha}{\sqrt{\pi}} \left\{ \int_{-\infty}^{\infty} e^{-\sigma} \sigma^{1+\alpha} d\sigma \right\} \frac{(t_1 - t_2)^{(\alpha+1)/2}}{1 + \alpha}$$

$$= C_2(\alpha) (t_1 - t_2)^{(\alpha+1)/2} \quad (A3)$$

where  $C_2(\alpha) = \frac{2^\alpha}{\sqrt{\pi}(1+\alpha)} \int_{-\infty}^{\infty} e^{\sigma} \sigma^{1+\alpha} d\sigma$

if  $x = 0$ ,

$$I_1 \leq \frac{1}{2} C_2(\alpha) (t_1 - t_2)^{(\alpha+1)/2} \quad (A4)$$

When  $j = 2$ ,

$$K_{xx}(x, y, t) = \frac{1}{2\sqrt{\pi} t} \left\{ e^{-(x-y)^2/4t} \left[ -\frac{1}{2t} + \frac{(x-y)^2}{4t^2} \right] \right. \\ \left. + \left[ -\frac{1}{2t} + \frac{(x+y)^2}{4t^2} \right] e^{-(x+y)^2/4t} \right\}$$

$$\begin{aligned}
I_2 &= \int_{t_2}^{t_1} \int_0^\infty |K_{xx}(x, y, t_1 - \tau)| |x - y|^\alpha dy d\tau \\
&\leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \frac{e^{-(x-y)^2/4(t_1-\tau)}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{(x-y)^2}{4|t_1-\tau|} \right] |x-y|^\alpha dy d\tau \\
&\quad + \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \frac{e^{-(x+y)^2/4(t_1-\tau)}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{(x+y)^2}{4|t_1-\tau|} \right] |x-y|^\alpha dy d\tau
\end{aligned}$$

Since  $|x - y| \leq |x + y|$

$$\begin{aligned}
I_2 &\leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_{-\infty}^\infty \frac{e^{-(x-y)^2/4(t_1-\tau)}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{|x-y|^2}{4|t_1-\tau|} \right] |x-y|^\alpha dy d\tau \\
&\leq \frac{2^\alpha}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \int_{t_2}^{t_1} |t_1 - \tau|^{(\alpha/2)-1} d\tau \\
&\leq \frac{C_3(\alpha)}{\alpha} (t_1 - t_2)^{\alpha/2}
\end{aligned} \tag{A5}$$

Where  $C_3(\alpha) = \frac{2^\alpha}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma$

if  $x = 0$ , then

$$I_2 \leq \frac{C_3(\alpha)}{2\alpha} (t_1 - t_2)^{\alpha/2} \tag{A6}$$

The estimate for

$$K_{xxt}(x, y, t) = \frac{1}{2\sqrt{\pi}} \left\{ e^{-(x-y)^2/4t} \left[ -\frac{6}{8} \frac{(x-y)^2}{t^{(7/2)}} + \frac{(x-y)^4}{16 t^{(9/2)}} + \frac{3}{3 t^{5/2}} \right] \right. \\ \left. + e^{-(x+y)^2/4t} \left[ -\frac{6}{8} \frac{(x+y)^2}{t^{(7/2)}} + \frac{(x+y)^4}{16 t^{(9/2)}} + \frac{3}{3 t^{5/2}} \right] \right\}$$

$$I_3 = \int_{t_2}^{t_1} \int_0^\infty |K_{xxt}(0, y, t_1 - \tau)| |y|^\alpha dy d\tau$$

$$\leq \frac{2^\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\sigma^2} \left[ 2\sigma^4 + 6\sigma^2 + (3/2) \right] \sigma^\alpha \int_{t_2}^{t_1} |t_1 - \tau|^{(\alpha/2)-2} d\tau \quad (A7)$$

$$\leq C_4(\alpha) \frac{(t_1 - t_2)^{(\alpha/2)-1}}{1 - (\alpha/2)} \quad (A8)$$

$$\text{where } C_4(\alpha) = \frac{2^\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\sigma^2} \sigma^\alpha \left[ 2\sigma^4 + 6\sigma^2 + (3/2) \right] d\sigma$$

(For complete calculation please see Appendix B (B8))

Let the value of  $I_4$  is estimated, where

$$I_4 = \int_{t_2}^{t_1} \int_0^\infty |K_t(x, y, t_1 - \tau)| |x-y|^\alpha dy d\tau$$

$$K_t(x, y, t) = \frac{1}{2\sqrt{\pi}} \left[ \frac{1}{\sqrt{t}} e^{-(x-y)^2/4t} \frac{(x-y)^2}{4t^2} - \frac{1}{2t^{(3/2)}} \times e^{-(x-y)^2/4t} \right. \\ \left. + \frac{1}{\sqrt{t}} e^{-(x+y)^2/4t} \frac{(x+y)^2}{4t^2} - \frac{1}{2t^{(3/2)}} e^{-(x+y)^2/4t} \right]$$

Now

$$\begin{aligned}
 I_4 &= \int_{t_2}^{t_1} \int_0^\infty |K_t(x, y, t_1 - \tau)| |x - y|^\alpha dy d\tau \\
 &\leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \frac{e^{-(x-y)^2/4|t_1-\tau|}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{|x-y|^2}{4|t_1-\tau|} \right] |x - y|^\alpha dy d\tau \\
 &\quad + \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_0^\infty \frac{e^{-(x+y)^2/4|t_1-\tau|}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{|x+y|^2}{4|t_1-\tau|} \right] |x - y|^\alpha dy d\tau
 \end{aligned}$$

We have made use of the inequality  $|x - y| < |x + y|$

So

$$I_4 \leq \frac{1}{2\sqrt{\pi}} \int_{t_2}^{t_1} \int_{-\infty}^\infty \frac{e^{-(x-y)^2/4|t_1-\tau|}}{|t_1-\tau|^{3/2}} \left[ \frac{1}{2} + \frac{|x-y|^2}{4|t_1-\tau|} \right] |x - y|^\alpha dy d\tau$$

Let  $\sigma = \frac{y - x}{2\sqrt{t_1 - \tau}}$  Then  $d\sigma = \frac{1}{2\sqrt{t_1 - \tau}} dy$

$$I_4 \leq \frac{2^\alpha}{2\sqrt{\pi}} \left( \int_{-\infty}^\infty e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \right) \int_{t_2}^{t_1} |t_1 - \tau|^{(\alpha/2)-1} d\tau$$

$$\leq \frac{C_5(\alpha)}{\alpha} (t_1 - t_2)^{(\alpha/2)} \quad (A9)$$

Where  $C_5(\alpha) = \frac{2^\alpha}{\sqrt{\pi}} \left( \int_{-\infty}^\infty e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \right)$

# APPENDIX-B

## COMPUTATION OF DOMAIN CONSTANTS

We will need to estimate terms of the form

$$I_j = \int_0^t \int_0^1 \left| \frac{\partial}{\partial x_j} K(x, y, t-\tau) \right| |x - 2n - y|^\alpha dy d\tau ;$$

for  $j = 0, 1, 2,$  and  $0 < \alpha \leq 1$

$$\text{Where } K(x, y, t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} + e^{-(x+y-2n)^2/4t} \right]$$

In case  $j = 0$  ( $x, y$  are positive)

$$\begin{aligned} I_0 &= \int_0^t \int_0^1 |K(x, y, t-\tau)| |x - 2n - y|^\alpha dy d\tau \\ &= \int_0^t \int_0^1 \left| \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} \left[ e^{-(x-y-2n)^2/4(t-\tau)} + e^{-(x+y-2n)^2/4(t-\tau)} \right] \right| |x - y - 2n|^\alpha dy d\tau \\ &\leq \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{1}{\sqrt{4\pi|t-\tau|}} \left[ e^{-(x-y-2n)^2/4(t-\tau)} + e^{-(x+y-2n)^2/4(t-\tau)} \right] |x - y - 2n|^\alpha dy d\tau \end{aligned}$$

since  $|x - y - 2n| \leq |x + y - 2n|$ ,  $n \geq 0$ ,

then  $e^{-(x-y-2n)^2/4(t-\tau)} \geq e^{-(x+y-2n)^2/4(t-\tau)}$

$$\text{so } I_0 \leq \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi|t-\tau|}} e^{-(x-y-2n)^2/4(t-\tau)} |x - y - 2n|^\alpha dy d\tau$$

$$\text{Let us consider, } \sigma = \frac{y - (x - 2n)}{2\sqrt{t - \tau}}, \quad t > \tau \quad \text{then} \quad d\sigma = \frac{1}{2\sqrt{t - \tau}} dy$$

Hence

$$I_0 \leq \int_0^t \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\frac{-(x-2n)}{2\sqrt{t-\tau}} = \theta_1}^{\frac{1-(x-2n)}{\sqrt{t-\tau}} = \theta_2} e^{-\sigma^2} \sigma^\alpha d\sigma \times 2^\alpha |t - \tau|^{\alpha/2} d\tau$$

$$\leq \frac{2^{\alpha+2}}{\sqrt{\pi}} \left( \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^\alpha d\sigma \right) \int_0^t |t - \tau|^{\alpha/2} d\tau$$

$$\leq \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^\alpha d\sigma \times \frac{t^{\alpha/2+1}}{(\alpha/2) + 1}$$

$$= C_1(x, n, \alpha) t^{\alpha/2+1} \quad (B1)$$

$$\text{Where } C_1(x, n, \alpha) = \frac{2^{\alpha+3}}{(\alpha + 2) \sqrt{\pi}} \left[ \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^\alpha d\sigma \right] \quad (B2)$$

When  $j = 1$  ;

$$I_1 = \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |x - 2n - y|^\alpha dy d\tau$$

where

$$K_x(x, y, t) = \sum_{n=-\infty}^{\infty} -\frac{1}{\sqrt{4\pi t}} \left[ \frac{(x - y - 2n)}{2t} \times e^{-(x-y-2n)^2/4t} + \frac{(x + y - 2n)}{2t} \times e^{-(x+y-2n)^2/4t} \right]$$

so

$$\begin{aligned} I_1 &= \int_0^t \int_0^1 |K_x(x, y, t-\tau)| |x - y - 2n|^\alpha dy d\tau \\ &= \int_0^t \int_0^1 \left| \sum_{n=-\infty}^{\infty} -\frac{1}{\sqrt{4\pi(t-\tau)}} \left[ \frac{(x - y - 2n)}{2(t-\tau)} e^{-(x-y-2n)^2/4(t-\tau)} + \frac{(x + y - 2n)}{2(t-\tau)} e^{-(x+y-2n)^2/4(t-\tau)} \right] \right| \times |x - 2n - y|^\alpha dy d\tau \\ &\leq \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{1}{\sqrt{4\pi|t-\tau|}} \left[ \frac{|x - y - 2n|}{2|t-\tau|} \times e^{-(x-y-2n)^2/4|t-\tau|} + \frac{|x + y - 2n|}{2|t-\tau|} \times e^{-(x+y-2n)^2/4|t-\tau|} \right] |x - 2n - y|^\alpha dy dt \end{aligned}$$

Since  $|(x - 2n) - y| \leq |(x - 2n) + y|$ , for  $n \geq 0$

this implies

$$e^{-|x-2n-y|/4|t-\tau|} \geq e^{-|x-2n+y|/4|t-\tau|}$$

Therefore, one can write  $I_1$  as

$$I_1 \leq \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{2}{\sqrt{4\pi|t-\tau|}} \left[ \frac{|x-2n-y|^{\alpha+1}}{2|t-\tau|} e^{-(x-y-2n)^2/4|t-\tau|} \right] dy d\tau$$

Similarly we denote  $\sigma = \frac{y - (x - 2n)}{2\sqrt{t-\tau}}$  for  $t > \tau$

Hence

$$I_1 \leq \int_0^t \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\frac{(x-2n)}{2\sqrt{t-\tau}} = \theta_1}^{\frac{1-(x-2n)}{2\sqrt{t-\tau}} = \theta_2} e^{-\sigma^2} \sigma^{\alpha+1} 2^{\alpha+1} \frac{|t-\tau|^{(\alpha+1)/2}}{2|t-\tau|} d\sigma d\tau$$

$$= \int_0^t \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^{\alpha+1} 2^{\alpha} |t-\tau|^{(\alpha-1)/2} d\sigma d\tau$$

$$I_1 \leq \left( \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^{\alpha+1} d\sigma \right) \int_0^t |t-\tau|^{(\alpha-1)/2} d\tau$$

$$\leq \left( \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^{\alpha+1} d\sigma \right) \frac{|t|^{(\alpha+1)/2}}{(\alpha+1)/2}$$

$$\leq \left( \frac{2^{\alpha+3}}{\sqrt{\pi}(\alpha+1)} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^{\alpha+1} d\sigma \right) t^{(\alpha+1)/2}$$

$$= C_2(x, n, \alpha) t^{(\alpha+1)/2} \quad (B3)$$

Where

$$C_2(x, n, \alpha) = \frac{2^{\alpha+3}}{\sqrt{\pi}(\alpha+1)} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \sigma^{\alpha+1} d\sigma \quad (B4)$$

Now for  $j = 2$ , one gets

$$I_2 = \int_0^t \int_0^1 \left| \frac{\partial}{\partial x^2} K(x, y, t-\tau) \right| |x - 2n - y|^\alpha dy d\tau$$

Integrating the Neumann function twice with respect to  $x$ , it becomes

$$K_{xx}(x, y, t-\tau) = \sum_{n=-\infty}^{\infty} -\frac{1}{\sqrt{4\pi t}} \left[ e^{-(x-y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x-y-2n)^2}{4t^2} \right) + e^{-(x+y-2n)^2/4t} \left( \frac{1}{2t} - \frac{(x+y-2n)^2}{4t^2} \right) \right]$$

Now substituting this value in  $I_2$ , it reduces to the form

$$\begin{aligned} I_2 &= \int_0^t \int_0^1 \left| \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} \left[ \frac{e^{-(x-2n-y)^2/4(t-\tau)}}{(t-\tau)} \times \right. \right. \\ &\quad \left. \left( -\frac{1}{2} + \frac{(x-y-2n)^2}{4(t-\tau)} \right) + \frac{e^{-(x-2n+y)^2/4(t-\tau)}}{(t-\tau)} \times \right. \\ &\quad \left. \left( -\frac{1}{2} + \frac{(x+y-2n)^2}{4(t-\tau)} \right) \right] \right| |x - 2n - y|^\alpha dy d\tau \\ &= \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{1}{\sqrt{4\pi|t-\tau|}} \left[ \frac{e^{-(x-2n-y)^2/4|t-\tau|}}{|t-\tau|} \times \right. \\ &\quad \left( \frac{1}{2} + \frac{|x-y-2n|^2}{4|t-\tau|} \right) + \frac{e^{-(x+y-2n)^2/4|t-\tau|}}{|t-\tau|} \times \\ &\quad \left( \frac{1}{2} + \frac{|x+y-2n|^2}{4|t-\tau|} \right) \left. \right] |x - 2n - y|^\alpha dy d\tau \end{aligned}$$

$$\begin{aligned}
& \leq \int_0^t \int_0^1 2 \sum_{n=0}^{\infty} \frac{2}{\sqrt{4\pi} |t-\tau|} \left[ \frac{e^{-(x-2n-y)^2/4|t-\tau|}}{|t-\tau|} \right. \\
& \quad \left. \left[ \frac{1}{2} + \frac{|x-y-2n|^2}{4|t-\tau|} \right] \right] |x-2n-y|^\alpha dy d\tau \\
& \leq \int_0^t \sum_{n=0}^{\infty} \frac{4}{\sqrt{\pi}} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \frac{1}{|t-\tau|} \sigma^\alpha 2^\alpha |t-\tau|^{\alpha/2} d\sigma d\tau \\
& = \left( \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \right) \int_0^t |t-\tau|^{(\alpha/2)-1} d\tau \\
& \leq \left( \frac{2^{\alpha+3}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \right) \frac{t^{\alpha/2}}{\alpha} \\
& = \frac{C_3(x, n, \alpha)}{\alpha} t^{\alpha/2}
\end{aligned} \tag{B5}$$

Where

$$C_3(x, n, \alpha) = \frac{2^{\alpha+3}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ \frac{1}{2} + \sigma^2 \right] \sigma^\alpha d\sigma \tag{B6}$$

Finally, we consider  $I_3$  where

$$I_3 = \int_0^t \int_0^1 |K_{\text{xx}t}(x, y, t-\tau)| |x-2n-y|^\alpha dy d\tau$$

$$\begin{aligned}
K_{\text{xx}}(x, y, t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{4\pi} t} \left[ e^{-(x-y-2n)^2/4t} \left( -\frac{1}{2t} + \frac{(x-y-2n)^2}{4t^2} \right) \right. \\
\left. + e^{-(x+y-2n)^2/4t} \left( -\frac{1}{2t} + \frac{(x+y-2n)^2}{4t^2} \right) \right]
\end{aligned}$$

or,

$$K_{xx}(x, y, t) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-y-2n)^2/4t} \times \left( -\frac{1}{2} t^{-3/2} + \frac{(x-y-2n)^2}{4} t^{-5/2} \right) + e^{-(x+y-2n)^2/4t} \left( -\frac{1}{2} t^{-3/2} + \frac{(x+y-2n)^2}{4} t^{-5/2} \right) \right]$$

so differentiating the above expression with respect to t,

$$K_{xxt}(x, y, t) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[ \frac{(x-y-2n)^2}{4} t^{-2} e^{-(x-2n-y)^2/4t} \times \left( -\frac{1}{2} t^{-3/2} + \frac{(x-y-2n)^2}{4} t^{-5/2} \right) + e^{-(x-2n-y)^2/4t} \left( \frac{3}{4} t^{-5/2} - \frac{(x-y-2n)^2}{4} \times \frac{5}{2} t^{-7/2} \right) + \frac{(x+y-2n)^2}{4} t^{-2} e^{-(x-2n+y)^2/4t} \left( -\frac{1}{2} t^{-3/2} + \frac{(x+y-2n)^2}{4} t^{-5/2} \right) + e^{-(x+y-2n)^2/4t} \left( \frac{1}{2} \times \frac{3}{2} t^{-5/2} - \frac{5}{2} \times \frac{(x+y-2n)^2}{4} t^{-7/2} \right) \right]$$

or

$$K_{xxt}(x, y, t) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[ \frac{(x-2n-y)^2}{4 t^2} \times e^{-(x-2n-y)^2/4t} \left( -\frac{1}{2 t^{3/2}} + \frac{(x-y-2n)^2}{4 t^{5/2}} \right) + e^{-(x-2n-y)^2/4t} \left( \frac{3}{4} t^{-5/2} - 5 \frac{(x-y-2n)^2}{8} t^{-7/2} \right) \right]$$

$$+ \frac{(x+y-2n)^2}{4t^2} e^{-(x-2n+y)^2/4t} \left( -\frac{1}{2} t^{-3/2} + \frac{(x+y-2n)^2}{4t^{5/2}} \right) \\ + e^{-(x+y-2n)^2/4t} \left( \frac{3}{4} t^{-5/2} - \frac{5(x+y-2n)^2}{8} t^{-7/2} \right) \Bigg]$$

Now arranging the terms properly,  $K_{xxt}(x,y,t)$  can be written as

$$K_{xxt}(x,y,t) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2n-y)^2/4t} \times \right. \\ \left( -\frac{(x-2n-y)^2}{8t^{7/2}} + \frac{(x-2n-y)^4}{16t^{9/2}} + \frac{3}{4} t^{-5/2} - \frac{5}{8} \times \frac{(x-y-2n)^2}{t^{7/2}} \right) \\ + e^{-(x-2n+y)^2/4t} \times \\ \left( -\frac{(x+y-2n)^2}{8t^{7/2}} + \frac{(x+y-2n)^4}{16t^{9/2}} + \frac{3}{4} t^{-5/2} - \frac{5}{8} \times \frac{(x+y-2n)^2}{t^{7/2}} \right) \Bigg]$$

Finally,  $K_{xxt}(x,y,t)$  is of the form

$$K_{xxt}(x,y,t) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \left[ e^{-(x-2n-y)^2/4t} \times \right. \\ \left( -\frac{6}{8} \frac{(x-2n-y)^2}{t^{7/2}} + \frac{(x-2n-y)^4}{16t^{9/2}} + \frac{3}{4} t^{-5/2} \right) \\ + e^{-(x-2n+y)^2/4t} \left( -\frac{6}{8} \frac{(x-2n+y)^2}{t^{7/2}} + \frac{(x-2n+y)^4}{16t^{9/2}} + \frac{3}{4} t^{-5/2} \right) \Bigg]$$

Now we estimate  $I_3$ , where

$$\begin{aligned}
 I_3 &= \int_0^t \int_0^1 |K_{xxt}(x, y, t-\tau)| |x - 2n - y|^\alpha dy d\tau \\
 &\leq \frac{4}{\sqrt{\pi}} \int_0^t \int_0^1 \sum_{n=0}^{\infty} \frac{1}{2} \left[ e^{-(x-2n-y)^2/4|t-\tau|} \left( \frac{6}{8} \times \frac{(x-2n-y)^2}{|t-\tau|^{7/2}} \right. \right. \\
 &\quad \left. \left. + \frac{3}{4} \frac{1}{|t-\tau|^{5/2}} + \frac{(x-2n-y)^4}{16|t-\tau|^{9/2}} \right) |x-y-2n|^\alpha dy d\tau
 \end{aligned}$$

For  $t > \tau$ , represent  $\sigma = \frac{y - (x - 2n)}{2\sqrt{t - \tau}}$  so  $d\sigma = \frac{1}{2\sqrt{t - \tau}} dy$

$$\text{and } \sigma^2 = \frac{(y - (x - 2n))^2}{4(t - \tau)} \quad \sigma^4 = \frac{(y - (x - 2n))^4}{16(t - \tau)^2}$$

Therefore,  $I_3$  becomes

$$\begin{aligned}
 I_3 &\leq \frac{4}{\sqrt{\pi}} \int_0^t \int_0^1 \sum_{n=0}^{\infty} e^{-(x-2n-y)^2/4(t-\tau)} \left[ \frac{1}{2(t-\tau)^{5/2}} \right. \\
 &\quad \left. \left( \frac{2}{16} \frac{(x-2n-y)^4}{(t-\tau)^2} + \frac{6}{4} \times \frac{(x-2n-y)^2}{(t-\tau)} + \frac{3}{2} \right) |x-y-2n|^\alpha dy d\tau
 \end{aligned}$$

Now substituting the values

$$\begin{aligned}
 I_3 &\leq \frac{4}{\sqrt{\pi}} \int_0^t \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ 6\sigma^2 + 2\sigma^4 + 3/2 \right] \sigma^\alpha 2^\alpha \frac{|t-\tau|^{\alpha/2}}{|t-\tau|^2} d\sigma d\tau \\
 &\leq \left( \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ 2\sigma^4 + 6\sigma^2 + 3/2 \right] \sigma^\alpha d\sigma \right) \int_0^t |t-\tau|^{(\alpha/2)-2} d\tau \quad (B7)
 \end{aligned}$$

$$\left( \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ 2\sigma^4 + 6\sigma^2 + 3/2 \right] \sigma^{\alpha} d\sigma \right) \frac{t^{(\alpha/2)-1}}{1 - \alpha/2}$$

$$C_4(x, n, \alpha) = \frac{t^{(\alpha/2)-1}}{1 - \alpha/2} \quad (B8)$$

here

$$C_4(x, n, \alpha) = \frac{2^{\alpha+2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\theta_1}^{\theta_2} e^{-\sigma^2} \left[ 2\sigma^4 + 6\sigma^2 + 3/2 \right] \sigma^{\alpha} d\sigma \quad (B9)$$

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